

Name : _____

Quiz: No. 7

Time: 15 minutes

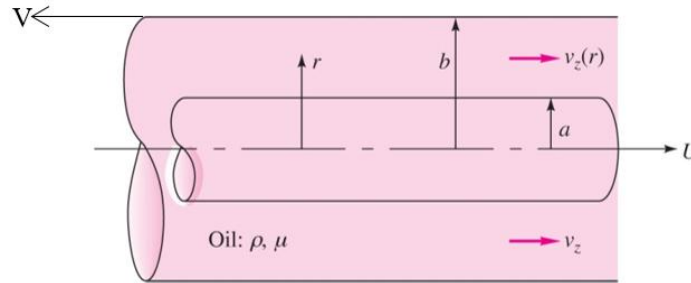
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Course: ME 5160, Fall 2021

The exam is closed book and closed notes.

The viscous oil in the Figure below is set into steady motion by a horizontal concentric inner cylinder moving axially to the right at constant velocity U inside an outer cylinder moving to the left at constant velocity V . Assume constant pressure and density, circumferentially symmetric flow, and a purely axial fluid motion ($v_r=v_\theta=0$). (a) Simplify the continuity and z-momentum equations and show that $r \frac{\partial v_z}{\partial r} = \text{constant}$. (b) Applying appropriate boundary conditions find the fluid velocity distribution $v_z(r)$.

(Hint: $\int \frac{1}{r} dr = \ln r$)



The equations of motion of an incompressible Newtonian fluid with constant density and viscosity in cylindrical coordinates (r, θ, z) with velocity components (v_r, v_θ, v_z) :

Continuity:

$$\frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

r-momentum:

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right) = \rho g_r - \frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r}(r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right]$$

θ -momentum:

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = \rho g_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r}(r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right]$$

z-momentum:

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right]$$

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Solution:FIND: $r \frac{\partial v_z}{\partial r} = \text{constant}$, $v_z(r)$

ASSUMPTIONS:

1. Steady flow ($\frac{\partial}{\partial t} = 0$)
2. Incompressible flow ($\rho = \text{constant}$)
3. Purely axial flow ($v_r = v_\theta = 0$)
4. Circumferentially symmetric flow, so properties do not vary with θ ($\frac{\partial}{\partial \theta} = 0$)
5. Constant pressure ($\partial p = 0$)
6. Horizontal cylinders ($g_z = 0$)

ANALYSIS:

(a)

Continuity:

$$\frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

$$0(3) + 0(3) + \frac{\partial v_z}{\partial z} = 0 \quad (1.5)$$

z-momentum:

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] \quad (0.5)$$

$$\rho(0(1) + 0(3) + 0(3,4) + 0(\text{continuity})) = 0(6) - 0(5) + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + 0(4) + 0(\text{continuity}) \right] \quad (1.5)$$

$$\frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) = 0$$

$$\frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) = 0$$

$$r \frac{\partial v_z}{\partial r} = \text{constant} \quad (1.5)$$

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(b)

Solve the above equation:

$$r \frac{\partial v_z}{\partial r} = C_1$$

$$\frac{\partial v_z}{\partial r} = \frac{C_1}{r}$$

$$v_z = C_1 \ln r + C_2 \quad (1)$$

Boundary conditions:

$$v_z(a) = U, \quad v_z(b) = -V \quad (1.5)$$

$$U = C_1 \ln a + C_2, \quad -V = C_1 \ln b + C_2$$

Find C_1 and C_2 :

$$U + V = C_1(\ln a - \ln b)$$

$$C_1 = \frac{(U + V)}{\ln \frac{a}{b}} \quad (1)$$

$$C_2 = U - \frac{(U + V)}{\ln \frac{a}{b}} \ln a \quad (1)$$

The final solution is:

$$v_z = \frac{(U + V)}{\ln \frac{a}{b}} \ln r + U - \frac{(U + V)}{\ln \frac{a}{b}} \ln a$$

$$= \frac{(U + V)}{\ln \frac{a}{b}} (\ln r - \ln a) + U$$

$$= \frac{(U + V)}{\ln \frac{a}{b}} \ln \frac{r}{a} + U \quad (0.5)$$