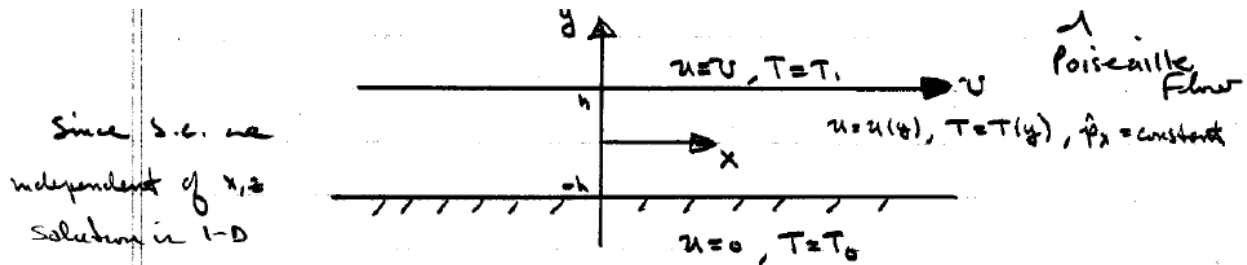


Appendix B: Steady Incompressible Flow Between Parallel Plates: Combined Couette and Poiseuille Flow With Heat Transfer.



$$\nabla \cdot \underline{V} = 0$$

$$u_x + v_y + w_z = 0$$

$$u_x = 0 \quad \text{i.e., fully developed flow}$$

$$\rho \frac{DV}{Dt} = -\nabla \hat{p} + \mu \nabla^2 \underline{V} \quad \frac{\partial u}{\partial t} + uu_x + vu_y + wu_z = 0$$

$$0 = -\hat{p}_x + \mu u_{yy}$$

$$\rho c_p \frac{DT}{Dt} = k \nabla^2 T + \Phi \quad \frac{\partial T}{\partial t} + uT_x + vT_y + wT_z = 0$$

$$\Phi = \tau_{ij} \frac{\partial u_i}{\partial x_j} = \mu (u_{i,j} + u_{j,i}) \frac{\partial u_i}{\partial x_j}$$

$$\begin{aligned} & \mu [2u_x^2 + 2v_y^2 + 2w_z^2 \\ & \quad + (v_x + u_y)^2 + (w_y + v_z)^2 + (u_z + w_x)^2] \\ & = \mu u_y^2 \end{aligned}$$

$$0 = kT_{yy} + \mu u_y^2$$

(Note inertia terms vanish identically and ρ is absent from equations)

Non-dimensional equations, but drop *

$$u^* = u/U \quad T^* = \frac{T-T_0}{T_1-T_0} \quad y^* = y/h$$

$$u_x = 0 \quad (1)$$

$$u_{yy} = \frac{h^2}{\mu U} \hat{p}_x = -B = \text{constant} \quad (2)$$

$$T_{yy} = \frac{\mu U^2}{\underbrace{k(T_1-T_0)}_{PrEc}} [-u_y^2] \quad (3)$$

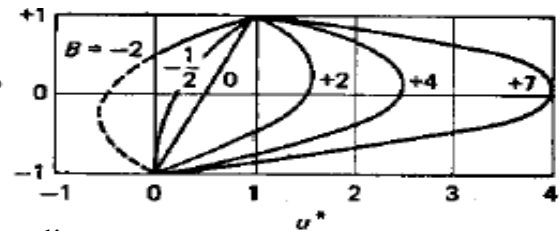
$$\begin{aligned} \text{B.C.} \quad y = 1 \quad u = 1 \quad T = 1 \\ y = -1 \quad u = 0 \quad T = 0 \end{aligned}$$

(1) is consistent with 1-D flow assumption. Simple form of (2) and (3) allow for solution to be obtained by double integration.

$$\Rightarrow u = \frac{1}{2}(1+y) + \frac{1}{2}B(1-y^2) \quad y=y/h$$

Linear flow
due to U

Parabolic flow
due to p_x



Note: linear
superposition since
 $\underline{V} \cdot \nabla \underline{V} = 0$

Solution depends on $B = -\frac{h^2}{\mu U} \hat{p}_x$ ($\hat{p}_x = \partial p / \partial x + \gamma \partial z / \partial x$)

$B < 0$ (adverse) $\hat{p}_x$ is opposite to U

$B < -0.5$ backflow occurs near lower wall

$|B| \gg 1$ flow approaches parabolic profile.

$$T = \frac{1}{2}(1 + y) + \frac{Pr E_c}{8}(1 - y^2) + \overbrace{\left(-\frac{Pr E_c B}{6}(y - y^3) + \frac{Pr E_c B^2}{12}(1 - y^4) \right)}^{\text{Pressure gradient effect}}$$

Pure
conduction

T rises due to
viscous

Dominant
term for $B \rightarrow \infty$

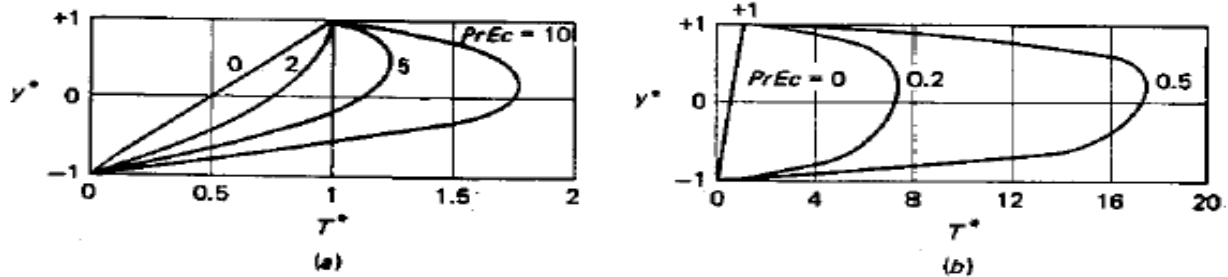


FIGURE 3-3
Temperature distributions for flow between parallel plates, Eq. (3-12): (a) pure Couette flow: $B = 0$; (b) mostly Poiseuille flow: $B = 20$.

Note: usually PrE_c is quite small

Substance	PrE_c	dissipation
Air	0.001	very small

Water	0.02
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$$Br = Pr E_c$$

$$= \text{Brinkman \#}$$

Crude oil	20	large
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Prandtl number $Pr = mC_p/k = \text{momentum diffusivity/thermal diffusivity}$

Eckert number $Ec = U^2/C_p(T_1 - T_0) = \text{advection transport/heat dissipation potential}$

$Br\# = \text{heat produced viscous dissipation/heat transported molecular conduction}$

Shear Stress

1) $\hat{p}_x = 0$ i.e., pure Couette Flow

$$B = -\frac{h^2}{\mu U} \hat{p}_x = 0$$

Using solution shown previously

$$u^* = \frac{1}{2}(1 + y^*) + \frac{1}{2}B(1 - y^{*2}) = \frac{1}{2}(1 + y^*)$$

Calculating wall shear stress

$$\frac{u}{U} = \frac{1}{2} \left(1 + \frac{y}{h} \right)$$

$$\frac{\partial \left(\frac{u}{U} \right)}{\partial \left(\frac{y}{h} \right)} = \frac{1}{2}$$

$$\tau_w = \mu \left. \frac{du}{dy} \right|_{y=-1} = \frac{\mu U}{2h}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{\frac{\mu U}{2h}}{\frac{1}{2}\rho U^2} = \frac{\mu}{\rho U h}$$

Since $Re_h = \rho U h / \mu$, $C_f = \frac{1}{Re_h}$

$P_0 = C_f Re = 1$: Better for non-accelerating flows since ρ is not in equations and $P_0 = \text{pure constant}$

2) $U = 0$ i.e. pure Poiseuille Flow

$$u^* = \frac{1}{2}B(1 - y^{*2}) \quad u_{y^*}^* = -By^* \quad u_y = -\frac{BU}{h^2}y \quad V_{ave} = \bar{u}$$

Where $B = \frac{-h^2}{\mu U} \hat{p}_x = \frac{2u_{max}}{U}$

Dimensional form $u = \underbrace{-\frac{1}{2} \frac{h^2}{\mu} \hat{p}_x}_{u_{max}} \left(1 - \left(\frac{y}{h}\right)^2\right) \quad Q = \int_{-h}^h u dy =$

$$\frac{4}{3}hu_{max}$$

$$\bar{u} = \frac{Q}{2h} = \frac{2}{3}u_{max} = V_{ave}$$

Remember that for laminar pipe flow, $V_{ave} = \frac{1}{2}u_{max}$

$$\tau_w = \mu u_y \big|_{y=\pm h} = -\mu \frac{BU}{h} \quad \text{upper}$$

$$= +\mu \frac{BU}{h} \quad \text{lower}$$

$$|\tau_w| = \mu \frac{BU}{h} = \mu \frac{2u_{max}}{h} = \mu 3 \bar{u} / h \quad \propto \bar{u} \quad \text{lam.}$$

$$\propto \rho \bar{u}^{-2} \quad \text{turb.}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{6\mu}{\rho \bar{u} h} = \frac{6}{Re_h} \quad \text{or} \quad P_0 = C_f Re_h = 6$$

Remember that for laminar pipe flow, $C_f = \frac{16}{Re_D}$ and $\tau_w = \frac{\mu 8 V_{ave}}{D}$,
i.e., except for numerical constants same functionality as for circular pipe.

Rate of heat transfer at the walls:

$$q_w = \left| k \frac{\partial T}{\partial y} \right|_{y \pm h} = \frac{k}{2h} (T_1 - T_0) \pm \mu \frac{U^2}{4h} \quad + = \text{upper, } - = \text{lower}$$

Heat transfer coefficient:

$$\zeta = \frac{q_w}{(T_1 - T_0)}$$

$$Nu = \frac{2h\zeta}{k} = 1 \pm Br/2$$

For $Br > 2$, both upper & lower walls must be cooled to maintain T_1 and T_0