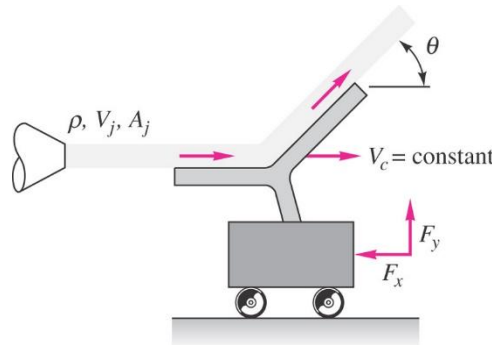


Appendix A: Example (use relative inertial reference frame):

A jet strikes a vane which moves to the right at constant velocity V_C on a frictionless cart. Compute (a) the force F_x required to restrain the cart and (b) the power P delivered to the cart. Also find the cart velocity for which (c) the force F_x is a maximum and (d) the power P is a maximum.



Solution:

Assume relative inertial coordinates with non-deforming CV i.e. CV moves at constant translational non-accelerating

$$\underline{V}_{CS} = u_{CS}\hat{i} + v_{CS}\hat{j} + w_{CS}\hat{k} = V_C\hat{i}$$

then $\underline{V}_R = \underline{V} - \underline{V}_{CS}$. Also assume steady flow $\underline{V} \neq \underline{V}(t)$ with $\rho = \text{constant}$ and neglect gravity effect.

Continuity:

$$0 = \rho \int_{CS} \underline{V}_R \cdot \underline{n} dA$$

$$-\rho V_{R1}A_1 + \rho V_{R2}A_2 = 0$$

$$V_{R1}A_1 = V_{R2}A_2 = \underbrace{(V_j - V_C)}_{V_{R1}=V_{Rx1}=V_j-V_C} A_j$$

Bernoulli without gravity:

$$p_1^0 + \frac{1}{2}\rho V_{R1}^2 = p_2^0 + \frac{1}{2}\rho V_{R2}^2$$

$$V_{R1} = V_{R2}$$

$$A_1 = A_2 = A_j$$

Momentum:

$$\Sigma \underline{F} = \rho \int_{CS} \underline{V}_R \underline{V}_R \cdot \underline{n} dA$$

$$\sum F_x = -F_x = \rho \int_{CS} V_{Rx} \underline{V}_R \cdot \underline{n} dA$$

$$-F_x = \rho V_{Rx1}(-V_{R1}A_1) + \rho V_{Rx2}(V_{R2}A_2)$$

$$\begin{aligned} -F_x &= \rho(V_j - V_C)[- (V_j - V_C)A_j] \\ &\quad + \rho(V_j - V_C) \cos \theta (V_j - V_C)A_j \end{aligned}$$

$$F_x = \rho(V_j - V_C)^2 A_j [1 - \cos \theta]$$

$$Power = V_C F_x = V_C \rho (V_j - V_C)^2 A_j (1 - \cos \theta)$$

$$F_{x_{max}} = \rho V_j^2 A_j (1 - \cos \theta), \quad V_C = 0$$

$$P_{max} \Rightarrow \frac{dP}{dV_C} = 0$$

$$P = V_C \rho (V_j^2 - 2V_C V_j + V_C^2) A_j (1 - \cos \theta)$$

$$= \rho(V_j^2 V_C - 2V_C^2 V_j + V_C^3) A_j (1 - \cos \theta)$$

$$\frac{dP}{dV_C} = \rho(V_j^2 - 4V_C V_j + 3V_C^2) A_j (1 - \cos \theta) = 0$$

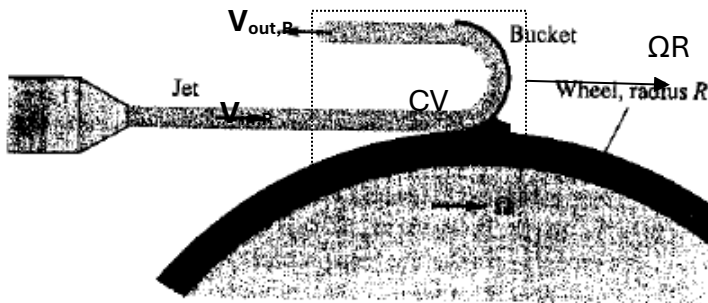
$$3V_C^2 - 4V_j V_C + V_j^2 = 0$$

$$V_C = \frac{+4V_j \pm \sqrt{16V_j^2 - 12V_j^2}}{6} = \frac{4V_j \pm 2V_j}{6}$$

$$\text{For } V_C = \frac{V_j}{3}: P_{max} = \frac{V_j}{3} \rho \left(\frac{2V_j}{3} \right)^2 A_j (1 - \cos \theta) = \frac{4}{27} V_j^3 \rho A_j (1 - \cos \theta)$$

Example (use absolute inertial and relative inertial reference frames)

P3.51 A liquid jet of velocity V_j and area A_j strikes a single 180° bucket on a turbine wheel rotating at angular velocity Ω ,



P3.51

as in Fig. P3.51. Derive an expression for the power P delivered to this wheel at this instant as a function of the system parameters. At what angular velocity is the maximum power delivered? How would your analysis differ if there were many, many buckets on the wheel, so that the jet was continually striking at least one bucket?

Assume gravity force is negligible, and the cross-section area of the jet does not change after striking the bucket. Taking moving CV at speed $\underline{V}_s = \Omega R \hat{i}$ enclosing jet and bucket:

Solution 1 (relative inertial coordinates)

Continuity: $-\dot{m}_{in,R} + \dot{m}_{out,R} = 0$

$$\dot{m}_R = \dot{m}_{in,R} = \dot{m}_{out,R} = \rho \int_{CS} \underline{V}_R \cdot \underline{n} \, dA$$

Bernoulli without gravity:

$$p_1^0 + \frac{1}{2} \rho V_{in,R}^2 = p_2^0 + \frac{1}{2} \rho V_{out,R}^2$$

$$V_{in,R} = V_{out,R}$$

Inlet $\underline{V_{in,R}} = (V_j - \Omega R)\hat{i}$

Outlet $\underline{V_{out,R}} = -(V_j - \Omega R)\hat{i}$

Since $-\rho V_{in,R}A_1 + \rho V_{out,R}A_2 = 0$

$$A_1 = A_2 = A_j$$

Momentum:

$$\sum F_X = -F_{bucket} = \dot{m}_R V_{out,R} - \dot{m}_R V_{in,R}$$

$$\begin{aligned} F_{bucket} &= -\dot{m}_R [-(V_j - \Omega R) - (V_j - \Omega R)] \\ &= 2\dot{m}_R (V_j - \Omega R) \\ &= 2\rho A_j (V_j - \Omega R)^2 \end{aligned}$$

$$\dot{m}_R = \rho A_j (V_j - \Omega R)$$

$$P = \Omega R F_{bucket} = 2\rho A_j \Omega R (V_j - \Omega R)^2$$

$$\frac{dP}{d\Omega} = 2\rho A_j R (V_j - \Omega R)^2 - 2\rho A_j \Omega R 2(V_j - \Omega R)R$$

$$= 2\rho A_j R [(V_j - \Omega R)^2 - 2R\Omega(V_j - \Omega R)]$$

$$= 2\rho A_j R (V_j - \Omega R) [V_j - \Omega R - 2R\Omega]$$

$$\frac{dP}{d\Omega} = 0 \rightarrow V_j - 3\Omega R = 0 \rightarrow \frac{V_j}{3} = \Omega R$$

$$P_{max} = 2\rho A_j \frac{V_j}{3} \left(V_j - \frac{V_j}{3}\right)^2 = 2\rho A_j \frac{V_j}{3} \frac{4V_j^2}{9} = \underbrace{\frac{8}{27}}_{0.296} \rho A_j V_j^3$$

If infinite number of buckets: $\dot{m}_R = \rho A_j V_j$ All jet mass flow

$$F_{bucket} = 2\rho A_j V_j (V_j - \Omega R) \quad \text{results in work.}$$

$$P = 2\rho A_j V_j \Omega R (V_j - \Omega R)$$

$$\frac{dP}{d\Omega} = 0 \quad \text{for} \quad \Omega R = \frac{V_j}{2} \quad P_{max} = \frac{1}{2} \rho A_j V_j^3$$

Solution 2 (absolute inertial coordinates)

$$\underline{V}_R = \underline{V} - \underline{V}_{CS} \rightarrow \underline{V} = \underline{V}_R + \underline{V}_{CS}$$

$$\underline{V}_{in} = V_j \hat{i}$$

$$\underline{V}_{out} = -(V_j - \Omega R) \hat{i} + \Omega R \hat{i} = -(V_j - 2\Omega R) \hat{i}$$

Continuity: from solution 1

$$-V_{in,R} + V_{out,R} = 0$$

express in the absolute inertial coordinates: $\underline{V}_R = \underline{V} - \underline{V}_{CS}$

$$-(V_j - \Omega R) \hat{i} + (V_j - 2\Omega R + \Omega R) \hat{i} = 0$$

Momentum:

$$\sum F_x = -F_{bucket} = \dot{m}(V_{out} - V_{in})$$

$$= \rho A_j (V_j - \Omega R) [-(V_j - 2\Omega R) - V_j]$$

$$F_{bucket} = 2\rho A_j (V_j - \Omega R)^2$$

Same as Solution 1.