Appendix A: Momentum Transport and Stress Tensor Overview

Momentum transport differs fundamentally from heat and mass transport because momentum is a vector ($\rho\mathbf{u}$), making its flux a second-order tensor. The stress tensor $\boldsymbol{\sigma}$ encapsulates both thermodynamic pressure (p) and viscous stresses ($\boldsymbol{\tau}$).

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau}$$

For a Newtonian fluid, the viscous stress tensor $\boldsymbol{\tau}$ is proportional to the symmetric part of the velocity gradient tensor:

$$\boldsymbol{\tau} = \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I}$$

Here, $\mu$is dynamic viscosity and $\lambda$is the second coefficient of viscosity.

Comparing Transport Fluxes

Transport Phenomenon	Transported Quantity	Gradient Driver	Governing Flux Law	Flux Type
Mass Transport	Concentration (C)	∇C	Fick's Law: $\mathbf{J} = -D\nabla C$	Vector
Heat Transport	Temperature (T)	∇T	Fourier's Law: $\mathbf{q} = -k\nabla T$	Vector
Momentum Transport	Velocity ($\mathbf{u}$)	$\nabla\mathbf{u}$	Newton's Law: $\boldsymbol{\tau} = \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$	Tensor (2nd-Order)

Viscous Term in Navier–Stokes as Momentum Diffusion

The incompressible momentum equation is given by:

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{f}$$

Dividing by density ρisolates the kinematic viscosity $\nu = \frac{\mu}{\rho}$:

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{g}$$

- **Role of Kinematic Viscosity (ν):** Kinematic viscosity acts as the momentum diffusivity coefficient (with dimensions m^2/s), direct counterpart to thermal diffusivity (α) and mass diffusivity (D).
- **Laplacian Operator ($\nabla^2\mathbf{u}$):** Represents local profile curvature. High localized velocity peaks ($\nabla^2\mathbf{u} < 0$) are smoothed out over time as momentum diffuses laterally into neighboring slower fluid layers.

Physical Mechanisms in Terms of Gradients

In all three transport phenomena, physical transport is driven by a spatial gradient—a measure of how much a physical quantity changes across space. The gradient acts as the thermodynamic "driving force" that moves the system toward equilibrium by diffusing the transported quantity from regions of higher potential to regions of lower potential.

1. Mass Transport: Driven by Concentration Gradient (∇C)

- **Mechanism:** Thermal agitation causes fluid molecules to move randomly in all directions. When a region has a high concentration of a specific species (C), more molecules randomly cross out of that region per unit time than enter it from neighboring lower-concentration regions.

- **Gradient Role:** The spatial gradient ∇Cdefines the net direction of molecular migration. Mass flux **J**acts down the concentration gradient (from high to low concentration) to eliminate the imbalance:

$$\mathbf{J} = -D\nabla C$$

2. Heat Transport: Driven by Temperature Gradient (∇T)

- **Mechanism:** Temperature represents the average kinetic energy of molecular translational, rotational, and vibrational motions. When adjacent fluid or solid regions are at different temperatures, faster-moving molecules collide with slower-moving neighbors, exchanging kinetic energy across the interface.

- **Gradient Role:** The temperature gradient ∇T establishes the directional slope of thermal energy. Heat flux $\mathbf{q}$ flows down the thermal gradient (from hot to cold) to homogenize thermal energy across the domain:

$$\mathbf{q} = -k\nabla T$$

3. Momentum Transport: Driven by Velocity Gradient ($\nabla\mathbf{u}$)

- **Mechanism:** Imagine fluid layers sliding past one another at different speeds. As molecules randomly hop across the boundary between layers due to thermal agitation, they carry their mean forward bulk velocity (and thus their vector momentum $\rho\mathbf{u}$) with them.

  - A fast-moving molecule entering a slower layer collides with local molecules, speeding the slower layer up.

  - A slow-moving molecule entering a faster layer act as drag, slowing the faster layer down.

- **Gradient Role:** The velocity gradient tensor $\nabla\mathbf{u}$ defines the shear deformation rate across adjacent fluid layers. The viscous stress tensor $\boldsymbol{\tau}$ transfers forward momentum across perpendicular surface normals to smooth out velocity differences:

$$\boldsymbol{\tau} = \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$$

Unified Transport Structure

At the microscopic level, all three processes follow the exact same gradient-driven transport law:

$$\text{Flux} = -(\text{Transport Coefficient}) \times (\text{Gradient of Transported Quantity})$$

Because mass and heat are scalar quantities, their driving forces ($\nabla C, \nabla T$) produce **vector fluxes ($\mathbf{J}, \mathbf{q}$)**. Because momentum is already a directional vector ($\rho\mathbf{u}$), its cross-stream transport by a velocity gradient requires two spatial directions—the momentum direction and the surface normal direction—yielding a **2nd-order tensor flux** ($\boldsymbol{\tau}$).

The negative sign in transport flux equations reflects the **second law of thermodynamics**: transport spontaneously occurs **down a gradient**, moving from regions of higher potential to lower potential to restore equilibrium.

Physical Meaning of the Minus Sign

When a quantity varies over space, its spatial gradient (∇) points in the direction of **steepest increase** (from low to high concentration/temperature).

Because physical fluxes naturally move in the opposite direction—from high to low—a negative sign must be explicitly inserted into the transport law so that the flux vector points in the actual physical direction of movement.

Mass, Heat, and Momentum Comparison

- **Mass Transport (Fick's Law):** $\mathbf{J} = -D\nabla C$
  - **Gradient (∇C):** Points toward *higher* concentration.
  - **Minus Sign:** Ensures net molecular movement **J**flows toward *lower* concentration to eliminate regional species imbalances.
- **Heat Transport (Fourier's Law):** $\mathbf{q} = -k\nabla T$
  - **Gradient (∇T):** Points toward *higher* temperature.
  - **Minus Sign:** Ensures thermal energy flux **q**flows from *hot to cold*.

- **Momentum Transport (Newton's Law):**
    - In scalar 1D shear flow, the flux of x-momentum in the y-direction is written as $\tau_{yx} = -\mu \frac{du_x}{dy}$.
    - **Gradient ($\frac{du_x}{dy}$):** Points toward *faster* fluid layers.
    - **Minus Sign:** Indicates momentum flows from fast-moving fluid layers into slower-moving layers to smooth out velocity differences.

Sign Convention Difference for Viscous Stress

In fluid mechanics, the viscous stress tensor $\boldsymbol{\tau}$ is often written without a negative sign ($\boldsymbol{\tau} = \mu \nabla \mathbf{u}$) depending on sign conventions:

1. **Thermodynamic / Transport Convention:** Flux $= -$(Diffusivity) $\times$ (Gradient). Here, $\boldsymbol{\tau}$ is defined as the *momentum flux* (momentum transferred per unit area per unit time across a surface), requiring the minus sign.

   Mechanical Stress Convention: $\boldsymbol{\tau}$ is defined as the *force per unit area* exerted by surrounding fluid. Because a faster layer pulls forward on a slower layer in the direction of positive motion, the viscous force definition absorbs the sign convention, rendering it positive $\boldsymbol{\tau} = \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$

Despite this convention choice for stress, the underlying diffusion term in the momentum balance retains its negative diffusive character: $-\nabla \cdot \boldsymbol{\tau} = -\nu \nabla^2 \mathbf{u}$, smoothing localized velocity peaks ($\nabla^2 \mathbf{u} < 0$) over time.