

Stability Nearly Parallel Viscous Flows

Assume 2D main/base flow for simplicity: $U(x, y), V(x, y)$ and nearly parallel flow, i.e., $V \ll U$ and $U_x \ll U_y$. With perturbation $\underline{v} = (u, v, w)$ and p .

$$\underline{\tilde{u}} = (U + u, v, w) \quad \tilde{p} = P + p$$

Both steady mean flow U, V, P and \underline{v}, p satisfy NS. All variables are nondimensional using U_∞, L, ρ .

momentum: total

$$u_t + (U + u)(U + u)_x + (V + v)(U + u)_y = -(P + p)_x + Re^{-1}\nabla^2(U + u)$$

$$v_t + (U + u)v_x + vu_y + wu_z = -(P + p)_y + Re^{-1}\nabla^2v$$

$$w_t + (U + u)w_x + vw_y + ww_z = -(P + p)_z + Re^{-1}\nabla^2w$$

momentum: main

$$UU_x + VU_y = -P_x + Re^{-1}\nabla^2U$$

$$-P_y = 0 \quad -P_z = 0$$

Expand total and linearize by neglecting products perturbation, V , and U_x ; and subtract mean:

$$u_t + Uu_x + vU_y = -p_x + Re^{-1}\nabla^2u$$

$$v_t + Uv_x = -p_y + Re^{-1}\nabla^2v \quad (1)$$

$$w_t + Uw_x = -p_z + Re^{-1}\nabla^2w$$

$$\text{Similarly for continuity: } \nabla \cdot \underline{v} = 0 \quad (2)$$

(1) + (2) linear system for \underline{v} and p for specified $U(y)$

Dependence main flow $f(x)$ suppressed by “nearly parallel” assumption. At any x , it is as if the main flow profile at this station continued upstream/downstream without changing.

For either channel flows or unbounded $\underline{v}(\pm h) = \underline{v}(\pm\infty) = 0$ and BC p are not required.

Assume normal mode solutions:

$$v_i = \hat{v}_i(y) \exp[i(\alpha x + \beta z - \alpha c_R t)] \exp(\alpha c_I t) \quad (3)$$

$$p = \hat{p}(y) \exp[i(\alpha x + \beta z - \alpha c_R t)] \exp(\alpha c_I t)$$

For wave numbers (α, β) real and specified Re , the substitution (3) into (1) + (2) is eigenvalue problem for $\hat{v}_i(y)$ and $\hat{p}(y)$ eigenfunctions for specific values $c =$ complex for temporal instability = eigenvalue $c = c_R + ic_I$.

$$i\alpha(U - c)\hat{u} + \hat{v}U_y = -i\alpha\hat{p} + Re^{-1}[\hat{u}_{yy} - (\alpha^2 + \beta^2)\hat{u}]$$

$$i\alpha(U - c)\hat{v} = -\hat{p}_y + Re^{-1}[\hat{v}_{yy} - (\alpha^2 + \beta^2)\hat{v}] \quad (4a,d)$$

$$i\alpha(U - c)\hat{w} = -i\beta\hat{p} + Re^{-1}[\hat{w}_{yy} - (\alpha^2 + \beta^2)\hat{w}]$$

$$i\alpha\hat{u} + i\beta\hat{w} + \hat{v}_y = 0$$

(4a-d) govern the viscous stability of a normal mode for any “nearly parallel” flow.

Squire's Theorem: for any unstable 3D disturbance there is a corresponding 2D disturbance ($\hat{w} = 0$) that is more unstable, i.e., the stability boundary of the flow with a 2D disturbance \hat{u} , \hat{v} is assured to be sufficient to find the lowest limit of linear stability. The following transformation is used:

$$\begin{aligned}\alpha^* &= (\alpha^2 + \beta^2)^{1/2}, & c^* &= c \\ \alpha^* u^* &= \alpha \hat{u} + \beta \hat{w}, & v^* &= \hat{v} \\ p^*/\alpha^* &= \hat{p}/\alpha, & \alpha^* Re^* &= \alpha Re \quad Re^{-1} = \frac{\alpha}{\alpha^*} Re^{*-1}\end{aligned}\tag{5}$$

Transform (4) using (6). 1st and 3rd (4) are added and 2nd and 4th simply transformed, as follows:

$$i\alpha^*(U - c^*)u^* + v^*U_y = -i\alpha^*p^* + Re^{*-1}(u_{yy}^* - \alpha^{*2}u^*)\tag{6}$$

$$i\alpha^*(U - c^*)v^* = -p_y^* + Re^{*-1}(v_{yy}^* - \alpha^*v^*)\tag{7}$$

$$i\alpha^*u^* + v_y^* = 0\tag{8}$$

$$(6) - (8) = (4) \text{ for } \beta = 0 \text{ and } \hat{w} = 0$$

Therefore, any solution 2D disturbance u^*, v^* with wave number α^* and speed $c^* =$ equivalent 3D with wave number α, β and speed c . Note temporal growth rate = $\exp(\alpha c_I t)$. Thus, for 3D disturbance equivalent 2D has large x -direction wave number since $\alpha^* = (\alpha^2 + \beta^2)^{1/2}$ and is more unstable $\alpha^* c_I^* > \alpha c_I$ (note $c_I^* = c_I$). Moreover, $Re^* = Re \alpha/\alpha^*$ i.e. $Re^* < Re$. Interest most unstable marginal stability curve \therefore Squire's Theorem shows that 2D disturbances produces this condition. Therefore use (4) with $\beta = 0$ and $\hat{w} = 0$.

$$i\alpha\hat{u} = i\alpha^*u^* - i\beta\hat{w} \quad \widehat{v}_y = v_y^*$$

$$i\alpha^*u^* + v_y^* = 0 \quad (4d) \rightarrow (8) \checkmark$$

$$i\alpha^*(U - c^*)v^* = -\frac{\alpha}{\alpha^*}p_y^* + \frac{\alpha}{\alpha^*}Re^{*-1}[v_{yy}^* - \alpha^{*2}v^*] \quad \div \alpha \text{ and } \times \alpha^*$$

$$i\alpha^*(U - c^*)v^* = -p_y^* + Re^{*-1}[v_{yy}^* - \alpha^{*2}v^*] \quad (4b) \rightarrow (7) \checkmark$$

$$i\alpha(U - c^*)\hat{u} + v^*U_y = -i\frac{\alpha^2}{\alpha^*}p^* + \frac{\alpha}{\alpha^*}Re^{*-1}[\widehat{u}_{yy} - \alpha^{*2}\hat{u}] \quad (4a)$$

$$i\alpha(U - c^*)\hat{w} = -i\beta\frac{\alpha}{\alpha^*}p^* + \frac{\alpha}{\alpha^*}Re^{*-1}[\widehat{w}_{yy} - \alpha^{*2}\hat{w}] \quad (4c)$$

$$i\alpha(U - c^*)\alpha\hat{u} + \alpha v^*U_y = -i\frac{\alpha^3}{\alpha^*}p^* + \frac{\alpha}{\alpha^*}Re^{*-1}[\alpha\widehat{u}_{yy} - \alpha^{*2}\hat{u}\alpha] \quad \times \alpha$$

$$i\alpha(U - c^*)\beta\hat{w} = -i\beta^2\frac{\alpha}{\alpha^*}p^* + \frac{\alpha}{\alpha^*}Re^{*-1}[\beta\widehat{w}_{yy} - \alpha^{*2}\hat{w}\beta] \quad \times \beta$$

Add

$$i\alpha(U - c^*)(\alpha\hat{u} + \beta\hat{w}) + \alpha v^*U_y = -i\alpha\frac{(\alpha^2 + \beta^2)}{\alpha^*}p^* + \frac{\alpha}{\alpha^*}Re^{*-1}[\alpha\widehat{u}_{yy} + \beta\widehat{w}_{yy} - \alpha^{*2}(\hat{u}\alpha + \beta\hat{w})]$$

$$i\alpha(U - c^*)\alpha^*u^* + \alpha v^*U_y = -i\alpha\alpha^*p^* + \frac{\alpha}{\alpha^*}Re^{*-1}[\alpha^*u_{yy}^* - \alpha^{*2}\alpha^*u^*]$$

$$i(U - c^*)\alpha^*u^* + v^*U_y = -i\alpha^*p^* + Re^{*-1}[u_{yy}^* - \alpha^{*2}u^*]$$

$$\alpha(4a) + \beta(4c) \rightarrow (6) \checkmark$$

Orr-Sommerfeld Equation

Based Squire's Theorem we can focus on 2D flow and use stream function such that continuity $\nabla \cdot \underline{v} = 0$ automatically satisfied.

$$u = \psi_y \quad v = -\psi_x$$

$\psi = \phi(y) \exp[i(\alpha x + \beta z - ct)]$ normal mode assumption with complex amplitude ϕ .

$$\hat{u} = \phi_y \quad \hat{v} = -i\alpha\phi \quad (9)$$

$$\text{Continuity: } i\alpha\phi_y + (-i\alpha\phi_y) = 0$$

A single equation for $\phi(y)$ is obtained by substituting (9) into (6), differentiating with respect to y so that p_y^* occurs, and then eliminating p_y^* in (7). (6) and (7) same (4a) and 4b) with $\beta = 0$ and $\hat{w} = 0$.

$$i\alpha(U - c)\phi_y - (i\alpha\phi)U_y = -i\alpha p + Re^{-1}(\phi_{yyy} - \alpha^2\phi_y)$$

$$\begin{aligned} i\alpha U_y \phi_y + i\alpha(U - c)\phi_{yy} - i\alpha\phi_y U_y - i\alpha\phi U_{yy} \\ = -i\alpha p_y + \frac{Re^{-1}}{i\alpha}(\phi_{yyyy} - \alpha^2\phi_{yy}) \end{aligned}$$

$$(U - c)\phi_{yy} - \phi U_{yy} = -p_y + \frac{1}{i\alpha Re}(\phi_{yyyy} - \alpha^2\phi_{yy})$$

$$i\alpha(U - c)(-i\alpha\phi) = -p_y + Re^{-1}(-i\alpha\phi_{yy} - \alpha^2(-i\alpha\phi))$$

$$\alpha^2(U - c)\phi = -p_y - \frac{i\alpha}{Re}(\phi_{yy} - \alpha^2\phi)$$

$$\begin{aligned}
& \alpha^2(U - c)\phi \\
&= (U - c)\phi_{yy} - \phi U_{yy} - \frac{1}{i\alpha Re} (\phi_{yyyy} - \alpha^2\phi_{yy}) \\
&\quad - \frac{i\alpha}{Re} (\phi_{yy} - \alpha^2\phi)
\end{aligned}$$

$$\begin{aligned}
& (U - c)[\phi_{yy} - \alpha^2\phi] - \phi U_{yy} \\
&= \frac{1}{i\alpha Re} (\phi_{yyyy} - \alpha^2\phi_{yy}) + \frac{i\alpha}{Re} (\phi_{yy} - \alpha^2\phi) \\
&= \frac{1}{i\alpha Re} (\phi_{yyyy} - \alpha^2\phi_{yy}) - \frac{1}{i\alpha Re} (\alpha^2\phi_{yy} - \alpha^4\phi) \\
&= \frac{1}{i\alpha Re} \left(\frac{d^2}{dy^2} - \alpha^2 \right)^2 \phi
\end{aligned}$$

$$(U - c)[\phi_{yy} - \alpha^2\phi] - \phi U_{yy} = \frac{1}{i\alpha Re} \left(\frac{d^2}{dy^2} - \alpha^2 \right)^2 \phi$$

$$\left(\frac{d^2}{dy^2} - \alpha^2 \right)^2 = \frac{d^4}{dy^4} - 2\alpha^2 \frac{d^2}{dy^2} + \alpha^4$$

OS with no-slip $\phi = \phi_y = 0$ at $\pm h$ for specified $U(y), Re$ and α is eigenvalue problem for eigenfunction $\phi(y)$ and eigenvalues $c = c_R + ic_I$. If BL flow that is unbounded also no-slip say at $y = 0$ and $\phi, \phi_y \rightarrow 0$ at $|y| = \infty$. For example, for given flow $c_R = c_R(\alpha, Re)$ and $c_I = c_I(\alpha, Re)$:

Unstable $c_I > 0$ and neutral stability for $c_I = 0$, i.e., $c_I = c_I(\alpha, Re) = 0$ provides curve of neutral stability. If c_I changes sign across $c_I = 0$ then also curve marginal stability, which separates stable and unstable regimes in α, Re space. Similarly for spatial stability with $\alpha = \alpha_R + i\alpha_I$ and $\alpha_I = 0$ etc.