Linear Stability and Transition in 1D Shear Flows

Contents

1 The Failure of Linear Stability Theory

Linear stability theory predicts the following for canonical shear flows:

- **Plane Couette flow:** linearly stable for all Re
- **Plane Poiseuille flow:** linearly unstable only above $Re \approx 5772$
- **Pipe flow (Hagen–Poiseuille):** linearly stable for all Re

Yet experiments show transition at $Re \sim 350$–1000 for Couette flow, $Re \sim 1000$ for Poiseuille flow, and $Re \sim 2300$ for pipe flow. The discrepancy is stark, and its resolution involves both linear and nonlinear mechanisms.

2 The Orr–Sommerfeld/Squire Decomposition

2.1 The v–η Formulation

After a Fourier transform in the homogeneous wall-parallel directions x and z,

$$\mathbf{q}(x, y, z, t) = \hat{\mathbf{q}}(y, t)\, e^{i(k_x x + k_z z)},$$

the linearised Navier–Stokes equations reduce to a coupled system for the wall-normal velocity v and wall-normal vorticity η:

$$\frac{d}{dt}\begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{OS} & 0 \\ \mathcal{L}_C & \mathcal{L}_{SQ} \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix}.$$

Here $\mathcal{L}_{OS}$ is the Orr–Sommerfeld operator (governing v), $\mathcal{L}_{SQ}$ is the Squire operator (governing η), and $\mathcal{L}_C \sim ik_x dU/dy$ is the one-way coupling from v to η.

2.2 Normal Mode Assumptions

The Fourier decomposition in x and z exploits translational symmetry and is *exact*—it is not an approximation. Crucially, **no assumption is made about the time dependence**: the system above is a well-posed initial-value problem in t.

The *additional* normal-mode assumption consists of writing $v, \eta \sim e^{\sigma t}$, which converts the evolution equations into an eigenvalue problem. This step discards transient dynamics and is precisely what fails to predict subcritical transition.

2.3 Squire's Theorem

Squire (1933) showed that for the purpose of finding the critical Reynolds number via modal analysis, it suffices to consider two-dimensional perturbations. The triangular structure of the operator above makes this decomposition clean. Ironically, however, it is the three-dimensional perturbations—the η component and its coupling from v—that are responsible for the largest transient growth via the lift-up effect.

3 Non-Normality and Transient Growth

3.1 Non-Normality of the Linearised Operator

The full v–η operator is **non-normal**: its eigenvectors are not orthogonal in the energy inner product,

$$\langle \phi_i, \phi_j \rangle_E \neq 0 \quad \text{for } i \neq j.$$

For a non-normal operator, all eigenvalues can have negative real part (the flow is modally stable) while the solution operator $\|e^{\mathcal{L}t}\|$ grows transiently and significantly before eventual decay.

3.2 The Role of Viscosity

Viscosity's role in non-normality is subtle:

1. The triangular (one-way coupled) structure of the operator exists in both viscous and inviscid cases and stems from the geometry of shear flow.
2. However, the *degree* of non-normality—how nearly parallel the eigenvectors are—depends on how different the eigenvalues of $\mathcal{L}_{OS}$ and $\mathcal{L}_{SQ}$ are. Viscosity causes these two operators to have significantly different eigenspectra, turning a mildly non-normal system into a severely non-normal one.
3. Additionally, viscosity sets a slow $O(Re)$ decay timescale for streamwise vortices, maximising the window for transient amplification.

The phase shift between v and η introduced by viscosity is a direct physical manifestation of eigenvector non-orthogonality.

3.3 Magnitude of Transient Growth

Mechanism	**Growth scaling**	**Timescale**
Orr mechanism	$O(1)$	$O(1/k_x U')$
Lift-up effect	$O(Re)$	$O(Re)$
Combined optimal	$O(Re^2)$	$O(Re)$

4 Physical Mechanisms

4.1 The Lift-Up Effect

Infinitesimal streamwise vortices (v, w perturbations) advect mean momentum $U(y)$ to generate large streamwise streaks (u perturbations). The growth is $O(Re)$ because:

- The forcing of u by v through $\mathcal{L}_C \sim ik_x dU/dy$ is $O(1)$.
- The vortices decay on a viscous timescale $O(Re)$.

- The streak amplitude therefore grows to $O(Re)$ before viscous decay.

4.2 The Orr Mechanism

Vortex lines initially tilted against the mean shear are rotated by the flow until they align with the shear direction, transiently amplifying the perturbation energy. This is an inviscid mechanism.

5 Role of Nonlinearity

Transient growth alone does not sustain turbulence; nonlinearity is required:

1. Linear transient growth amplifies small perturbations to $O(1)$ amplitude.
2. Nonlinear interactions trigger secondary instabilities of the streaks (inflectional or Kelvin–Helmholtz type).
3. These feed energy back into streamwise vortices via the **self-sustaining process** (Waleffe 1997), creating a subcritical bifurcation.

This *bypass transition* scenario (Morkovin) operates entirely without modal instability.

6 The Pseudospectrum Perspective

Trefethen et al. (1993) recast the problem in terms of the ε-pseudospectrum of $\mathcal{L}$:

$$\sigma_\varepsilon(\mathcal{L}) = \{\, z \in \mathbb{C} : \|(zI - \mathcal{L})^{-1}\| \geq \varepsilon^{-1} \,\}.$$

Even when all eigenvalues of $\mathcal{L}$ lie in the stable half-plane, the pseudospectrum can extend far into the unstable half-plane, meaning the resolvent norm is large and the system is highly sensitive to perturbations. This sensitivity, combined with nonlinear feedback, explains transition without requiring modal instability.

7 Summary

Mechanism	**Role**
Modal instability (eigenvalues)	Absent or insufficient for most shear flows
Non-normality / transient growth	Key linear mechanism; amplifies perturbations by $O(Re^2)$
Viscosity	Separates OS and Squire eigenspectra; increases transient growth as $Re \uparrow$
Nonlinearity	Essential for sustaining turbulence and subcritical bifurcations

8 Key References

- Orr, W.M.F. (1907). The stability or instability of the steady motions of a perfect liquid and of a viscous liquid. *Proc. R. Irish Acad.*
- Squire, H.B. (1933). On the stability of three-dimensional disturbances of viscous flow. *Proc. R. Soc. Lond. A*, 142, 621–628.
- Ellingsen, T. & Palm, E. (1975). Stability of linear flow. *Phys. Fluids*, 18, 487.
- Landahl, M.T. (1980). A note on an algebraic instability of inviscid parallel shear flows. *J. Fluid Mech.*, 98, 243–251.
- Butler, K.M. & Farrell, B.F. (1992). Three-dimensional optimal perturbations in viscous shear flow. *Phys. Fluids A*, 4, 1637–1650.
- Trefethen, L.N., Trefethen, A.E., Reddy, S.C. & Driscoll, T.A. (1993). Hydrodynamic stability without eigenvalues. *Science*, 261, 578–584.
- Waleffe, F. (1997). On a self-sustaining process in shear flows. *Phys. Fluids*, 9, 883–900.
- Schmid, P.J. & Henningson, D.S. (2001). *Stability and Transition in Shear Flows.* Springer.