Example Viscous-Nonlinear Destabilization

The integrated kinetic energy of the disturbance is obtained by subtracting the base flow from the the disturbed flow and integrating over a stationary control volume V having streamwise control surfaces chosen to coincide with the walls where the no-slip conditions are satisfied and having length (in streamwise direction) that is an integer number of the disturbance wave lengths.

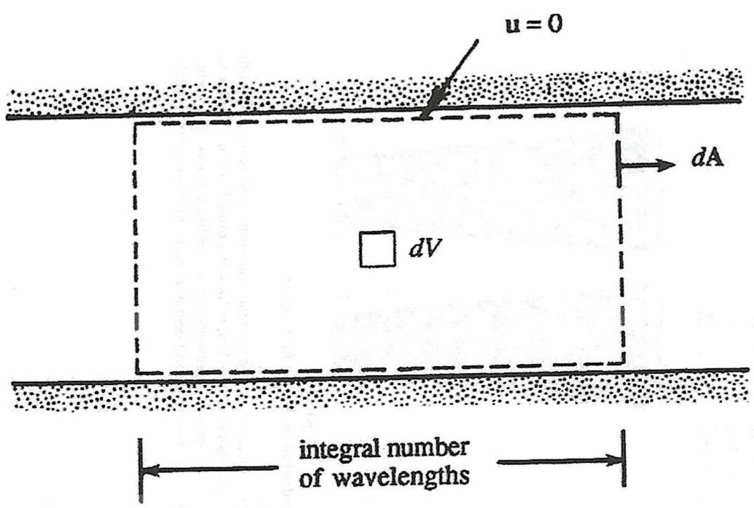

FIGURE 11.25 A control volume for deriving (11.88). Here, there is zero net flux across boundaries. This control volume can be extended to boundary-layer flow stability, when the boundary layer forms on the lower wall, by placing the upper control surface far enough from the lower wall so that the disturbance velocity $u_i \to 0$ on this control surface, even if this control surface may not abut the upper wall.

The derivation is provided below and should be compared with the derivation of the TKE equation ME:7268 Chapter 3 Part 3 pp. 4-8.

$$\frac{d}{dt}\int \frac{1}{2}u_i^2 dV = -\int u_i u_j \frac{\partial U_i}{\partial x_j} dV - \Lambda,$$

$$\Lambda = \nu \int (\partial u_i / \partial x_j)^2 dV$$

Is the total viscous dissipation rate of kinetic energy in V.

For 2D disturbance in a shear flow $\underline{U} = [U(y), 0,0]$ the disturbance energy equation becomes:

$$\frac{d}{dt}\int \frac{1}{2}\left(u^2 + v^2\right) dV = -\int uv\frac{\partial U}{\partial y} dV - \Lambda,$$

LHS = rate of change of KE of the disturbance

RHS first term = rate of production of the 2D disturbance energy by the interaction of the product of uv (Renolds shear stress) and mean shear U_y.

The value of uv averaged over a period is zero if u and v are out of phase, as is the case for modal analysis, which is consistent with inviscid stability analysis for profiles without inflection points.

However, for viscous 1D shear flows u and v are correlated, i.e., in phase producing the Reynolds shear stress, as per ME:7268 Chapter 3 Part 2 pp. 4-6. <uv> is < 0 and U_y is > 0; thus, the production is > 0 and larger than Λ such that the disturbance grows due to viscous induced nonlinearities.

The elegance is that the same equation contains both the stabilizing and destabilizing effects of viscosity, and the competition between them determines whether transition occurs.

This provides a simple example of viscous-nonlinear destabilization, which is actually a very complex process, as per Appendix B.

Exercise 11.14. Derive (11.88) starting from the incompressible Navier-Stokes momentum equation for the disturbed flow:

$$\frac{\partial}{\partial t}(U_i+u_i)+(U_j+u_j)\frac{\partial}{\partial x_j}(U_i+u_i)=-\frac{1}{\rho}\frac{\partial}{\partial x_i}(P+p)+\nu\frac{\partial^2}{\partial x_j\partial x_j}(U_i+u_i), \qquad (11.96)$$

where U_i and u_i represent the basic flow and the disturbance, respectively. Subtract the equation of motion for the basic state from (11.96), multiply by u_i and integrate the result within a stationary volume having stream-wise control surfaces chosen to coincide with the walls where no-slip conditions are satisfied or where $u_i \rightarrow 0$, and having a length (in the stream-wise direction) that is an integer number of disturbance wavelengths.

Solution 11.14. The basic flow momentum equation is: $\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\frac{\partial^2 U_i}{\partial x_j\partial x_j}$.

Subtracting this from (11.96) produces: $\frac{\partial u_i}{\partial t}+u_j\frac{\partial U_i}{\partial x_j}+U_j\frac{\partial u_i}{\partial x_j}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}$.

Multiply by u_i and integrate each term over the CV.

$$\int_{CV}\left\{u_i\frac{\partial u_i}{\partial t}+u_iU_j\frac{\partial u_i}{\partial x_j}+u_iu_j\frac{\partial u_i}{\partial x_j}+u_iu_j\frac{\partial U_i}{\partial x_j}=-\frac{u_i}{\rho}\frac{\partial p}{\partial x_i}+u_i\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}\right\}dV. \qquad (\%)$$

Consider each term individually.

$$\int_{CV}u_i\frac{\partial u_i}{\partial t}dV=\int_{CV}\frac{\partial}{\partial t}\left(\frac{1}{2}u_i^2\right)dV=\frac{d}{dt}\int_{CV}\frac{1}{2}u_i^2dV.$$

Here the final equality follows because the CV is stationary and the volume integration leaves no spatial dependence.

$$\int_{CV}u_iU_j\frac{\partial u_i}{\partial x_j}dV=\int_{CV}\left(\frac{1}{2}U_j\frac{\partial u_i^2}{\partial x_j}+u_i^2\frac{\partial U_j}{\partial x_j}\right)dV=\int_{CV}\left(\frac{1}{2}\frac{\partial(u_i^2U_j)}{\partial x_j}\right)dV=\int_{CS}\frac{1}{2}u_i^2U_jn_jdA=0$$

$$\int_{CV}u_iu_j\frac{\partial u_i}{\partial x_j}dV=\int_{CV}\left(\frac{1}{2}u_j\frac{\partial u_i^2}{\partial x_j}+u_i^2\frac{\partial u_j}{\partial x_j}\right)dV=\int_{CV}\left(\frac{1}{2}\frac{\partial(u_i^2u_j)}{\partial x_j}\right)dV=\int_{CS}\frac{1}{2}u_i^2u_jn_jdA=0$$

Here *CS* is the control surface, and n_j are the components of the outward normal vector from the control surface. And, $\partial u_j/\partial x_j = 0 = \partial U_j/\partial x_j$ has been used along with Gauss' theorem, and the fact that velocity fluctuations are zero on the stream wise control surfaces (no slip), or equal on the stream-normal control surfaces where n_j changes sign between the inlet and the outlet (periodic) leading to cancellation.

$$\int_{CV}u_i\frac{\partial p}{\partial x_i}dV=\int_{CV}\left(\frac{\partial(pu_i)}{\partial x_i}+p\frac{\partial u_i}{\partial x_i}\right)dV=\int_{CS}pu_in_idA=0.$$

$$\int_{CV}u_i\frac{\partial u_i}{\partial x_j^2}dV=\int_{CV}\frac{\partial}{\partial x_j}\left(u_i\frac{\partial u_i}{\partial x_j}\right)dV-\int_{CV}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}dV=\int_{CS}u_i\frac{\partial u_i}{\partial x_j}n_jdA-\int_{CV}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}dV=-\int_{CV}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}dV.$$

Here again incompressibility, Gauss' theorem, and the no slip and periodic boundary conditions lead to the simplifications. Thus, (%) becomes:

$$\frac{d}{dt}\int_{CV}\frac{1}{2}u_i^2dV=-\rho\int_{CV}u_iu_j\frac{\partial U_i}{\partial x_j}dV-\nu\int_{CV}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}dV,$$

which is (11.88).