- The Rayleigh equation:

$$(U-c)(v''-\alpha^2 v) - U''v = 0. \qquad (38)$$

- Rayleigh's inflexion point theorem.
- Divide (38) by $(U-c)$, multiply by $\bar{v}$ (the complex conjugate of v) and integrate across the flow domain:

$$\int_{y_1}^{y_2} \bar{v}v'' - \left(\frac{U''}{U-c} + \alpha^2\right)|v|^2 \, \mathrm{d}y = 0. \qquad (39)$$

- Integrate the first term by parts:

$$\left[\bar{v}v'\right]_{y_1}^{y_2} + \int_{y_1}^{y_2} -\bar{v}'v' - \left(\frac{U''}{U-c} + \alpha^2\right)|v|^2 \, \mathrm{d}y \quad = \quad 0 \qquad (40)$$

$$\Rightarrow \quad \int_{y_1}^{y_2} |v'|^2 + \alpha^2|v|^2 \, \mathrm{d}y + \int_{y_1}^{y_2} \frac{U''|v|^2}{U-c} \, \mathrm{d}y \quad = \quad 0. \qquad (41)$$

- Consider the imaginary part of (41), taking α real, with c possibly complex:

$$c_i \int_{y_1}^{y_2} \frac{U''|v|^2}{|U-c|^2} \, \mathrm{d}y = 0. \qquad (42)$$

- Therefore, for instability, i.e. $c_i \neq 0$, the integral in (42) must be zero, which can only be true if either $U'' \equiv 0$, or U'' changes sign at least once in the interval $y_1 < y < y_2$.
- This proves Rayleigh's (1880) inflexion point theorem:

  A necessary, but not sufficient, condition for instability is that the velocity profile have an inflexion point.

Fjørtoft's theorem

- Let there be an inflexion point at $y = y_I$, and let $U_I = U(y_I)$.
- Note that if $c_i \neq 0$, then (42) implies

$$(c_r - U_I) \int_{y_1}^{y_2} \frac{U''|v|^2}{|U - c|^2} \, \mathrm{d}y = 0. \qquad (43)$$

- The real part of (41) is

$$\int_{y_1}^{y_2} \frac{U''(U - c_r)|v|^2}{|U - c|^2} \, \mathrm{d}y = - \int_{y_1}^{y_2} |v'|^2 + \alpha^2 |v|^2 \, \mathrm{d}y. \qquad (44)$$

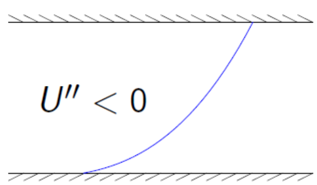
- Adding (43) and (44) leads to

$$\int_{y_1}^{y_2} \frac{U''(U - U_I)|v|^2}{|U - c|^2} \, \mathrm{d}y < 0. \qquad (45)$$

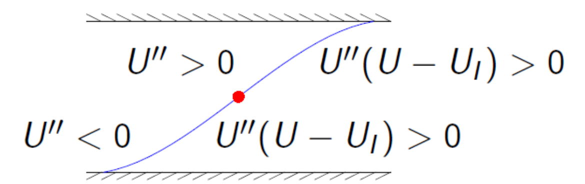
Equation (45) proves Fjørtoft's (1950) theorem:

A necessary, but not sufficient, condition for instability is that $U''(U - U_I) < 0$ somewhere in the flow.

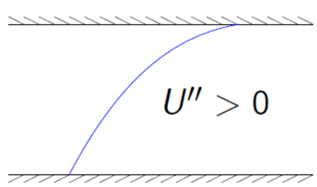
Examples of Rayleigh's and Fjørtoft's results:



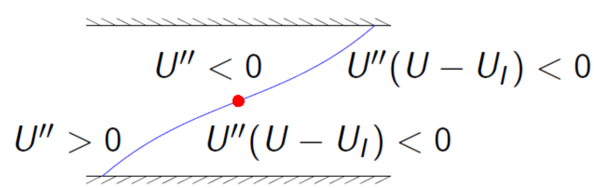
Stable by Rayleigh



Stable by Fjørtoft



Stable by Rayleigh



Could be unstable