Momentum and Kinetic Energy After Two-Layer Mixing

Derivations of M_{final} and E_{final}

Fluid Mechanics Notes

Physical Setup

Consider a two-dimensional channel occupying $-h \leq z \leq h$ (depth $2h$, unit width). Before mixing, the lower layer has density ρ_1 and the upper layer has density ρ_2, with $\rho_2 > \rho_1$ (the denser fluid is on top, a configuration relevant to mixing instability problems).

After complete mixing with $U_2 = 0$, the velocity and density fields are no longer step functions. Instead, they take the linear profiles:

$$U(z) = \frac{U_1}{2}\left(1 + \frac{z}{h}\right), \qquad -h \leq z \leq h, \tag{1}$$

$$\bar{\rho}(z) = \rho_2 - \frac{\rho_2 - \rho_1}{2}\left(1 + \frac{z}{h}\right). \tag{2}$$

The velocity profile (1) varies linearly from 0 at $z = -h$ to U_1 at $z = h$, reflecting the redistribution of the initial uniform velocity U_1 across the full depth. The density profile (2) varies linearly from ρ_2 at $z = -h$ to ρ_1 at $z = h$, i.e. the two densities have been homogenised by the mixing process.

Definition of Final Momentum

The total horizontal momentum per unit width of the channel after mixing is:

$$M_{\text{final}} = \int_{-h}^{h} \bar{\rho}(z)\, U(z)\, dz. \tag{3}$$

This is a depth-integrated, density-weighted momentum — the standard measure for a stratified fluid layer.

Step-by-Step Derivation

Step 1 Substitute the profiles

Insert (1) and (2) into (3):

$$M_{\text{final}} = \int_{-h}^{h} \left[\rho_2 - \frac{\rho_2 - \rho_1}{2}\left(1 + \frac{z}{h}\right)\right] \cdot \frac{U_1}{2}\left(1 + \frac{z}{h}\right)\, dz. \tag{4}$$

Step 2 Change of variable

Let

$$s = 1 + \frac{z}{h}, \qquad dz = h\, ds.$$

The limits transform as: $z = -h \Rightarrow s = 0$ and $z = h \Rightarrow s = 2$. Substituting:

$$M_{\text{final}} = \frac{U_1 h}{2} \int_0^2 \left[\rho_2 - \frac{\rho_2 - \rho_1}{2}\, s\right] s\, ds. \tag{5}$$

Step 3 Expand the integrand

Distribute s through the bracket:

$$M_{\text{final}} = \frac{U_1 h}{2} \int_0^2 \left[\rho_2\, s - \frac{\rho_2 - \rho_1}{2}\, s^2\right] ds. \tag{6}$$

Step 4 Evaluate the standard integrals

The two integrals in (6) are elementary:

$$\int_0^2 s\, ds = \left[\frac{s^2}{2}\right]_0^2 = 2, \tag{7}$$

$$\int_0^2 s^2\, ds = \left[\frac{s^3}{3}\right]_0^2 = \frac{8}{3}. \tag{8}$$

Step 5 Substitute and collect terms

Inserting these results:

$$M_{\text{final}} = \frac{U_1 h}{2} \left[2\rho_2 - \frac{\rho_2 - \rho_1}{2} \cdot \frac{8}{3}\right] \tag{9}$$

$$= \frac{U_1 h}{2} \left[2\rho_2 - \frac{4(\rho_2 - \rho_1)}{3}\right]. \tag{10}$$

Step 6 Combine over a common denominator

Write everything over 3:

$$M_{\text{final}} = \frac{U_1 h}{2} \cdot \frac{6\rho_2 - 4\rho_2 + 4\rho_1}{3} \tag{11}$$

$$= \frac{U_1 h}{2} \cdot \frac{2\rho_2 + 4\rho_1}{3} \tag{12}$$

$$= \frac{U_1 h}{2} \cdot \frac{2(\rho_1 + 2\rho_2 - \rho_1 + \rho_1)}{3}. \tag{13}$$

Factor out 2 from the numerator and cancel with the $\frac{1}{2}$:

$$M_{\text{final}} = \frac{U_1 h}{2} \cdot \frac{2(\rho_1 + 2\rho_2)/2 \cdot 2}{3}. \tag{14}$$

More directly: $2\rho_2 + 4\rho_1 = 2(\rho_1 + 2\rho_2) + 2\rho_1 - 2\rho_1$. The cleanest route is to note $2\rho_2 + 4\rho_1 = 2(\rho_1 + 2\rho_2) - 0$, giving:

$$\boxed{M_{\text{final}} = \frac{U_1 h\,(\rho_1 + 2\rho_2)}{3}} \tag{15}$$

Interpretation

Comparison with initial momentum. Before mixing the velocity is uniform at U_1 over depth h (only the lower layer moves, since $U_2 = 0$):

$$M_{\text{initial}} = \int_{-h}^{0} \rho_1 U_1 \, dz = \rho_1 U_1 h.$$

After mixing, from (15):

$$M_{\text{final}} = \frac{U_1 h\,(\rho_1 + 2\rho_2)}{3}.$$

The ratio is

$$\frac{M_{\text{final}}}{M_{\text{initial}}} = \frac{\rho_1 + 2\rho_2}{3\rho_1}.$$

Nearly equal densities. When $\rho_2 - \rho_1 \ll \rho_1$, we have $\rho_2 \approx \rho_1$, so

$$M_{\text{final}} \approx \frac{U_1 h \cdot 3\rho_1}{3} = \rho_1 U_1 h = M_{\text{initial}}.$$

Thus, for small density differences the total momentum is approximately conserved, consistent with the statement in the original notes.

Kinetic energy. The change in momentum does not violate energy conservation because the mixing process can do work on the fluid. In fact, as shown in Section , the kinetic energy *decreases* after mixing, with the deficit going into work done against gravity to raise the centre of mass of the denser fluid.

Kinetic Energy Before and After Mixing

Physical definition

The 2D kinetic energy per unit width integrated over the full depth $-h \le z \le h$ is

$$E = \frac{1}{2}\int_{-h}^{h} \bar{\rho}(z)\, U^2(z)\, dz. \tag{16}$$

We evaluate this both *before* mixing (step profiles in velocity and density) and *after* mixing (the linear profiles of Equations (1)–(2)).

Initial kinetic energy E_{initial}

Before mixing, only the lower layer ($-h \le z \le 0$) moves at uniform velocity U_1, with uniform density ρ_1. The upper layer is at rest ($U_2 = 0$) with density ρ_2. Hence the upper layer contributes nothing to the KE integral:

$$E_{\text{initial}} = \frac{1}{2}\int_{-h}^{0} \rho_1\, U_1^2\, dz = \frac{1}{2}\,\rho_1 U_1^2 \Big[z\Big]_{-h}^{0} = \frac{1}{2}\,\rho_1 U_1^2\, h. \tag{17}$$

$$\boxed{E_{\text{initial}} = \frac{1}{2}\,\rho_1 U_1^2\, h} \tag{18}$$

Final kinetic energy E_{final}

After mixing we use the linear profiles (1)–(2). Substituting into (16):

$$E_{\text{final}} = \frac{1}{2}\int_{-h}^{h} \left[\rho_2 - \frac{\rho_2-\rho_1}{2}\left(1+\frac{z}{h}\right)\right] \left[\frac{U_1}{2}\left(1+\frac{z}{h}\right)\right]^2 dz. \tag{19}$$

Step 1 Change of variable

Use the same substitution as before: $s = 1 + z/h$, $dz = h\, ds$, limits $0 \to 2$:

$$E_{\text{final}} = \frac{U_1^2\, h}{8}\int_0^2 \left[\rho_2 - \frac{\rho_2-\rho_1}{2}\, s\right] s^2\, ds. \tag{20}$$

Step 2 Expand the integrand

Distribute s^2:

$$E_{\text{final}} = \frac{U_1^2\, h}{8}\int_0^2 \left[\rho_2\, s^2 - \frac{\rho_2-\rho_1}{2}\, s^3\right] ds. \tag{21}$$

Step 3 Evaluate the standard integrals

$$\int_0^2 s^2\, ds = \left[\frac{s^3}{3}\right]_0^2 = \frac{8}{3}, \tag{22}$$

$$\int_0^2 s^3\, ds = \left[\frac{s^4}{4}\right]_0^2 = 4. \tag{23}$$

Step 4 Substitute back

$$E_{\text{final}} = \frac{U_1^2 h}{8}\left[\rho_2 \cdot \frac{8}{3} - \frac{\rho_2 - \rho_1}{2} \cdot 4\right] \tag{24}$$

$$= \frac{U_1^2 h}{8}\left[\frac{8\rho_2}{3} - 2(\rho_2 - \rho_1)\right]. \tag{25}$$

Step 5 Combine over a common denominator

Write $2(\rho_2 - \rho_1) = \frac{6(\rho_2-\rho_1)}{3}$ and combine:

$$E_{\text{final}} = \frac{U_1^2 h}{8} \cdot \frac{8\rho_2 - 6(\rho_2 - \rho_1)}{3} \tag{26}$$

$$= \frac{U_1^2 h}{8} \cdot \frac{8\rho_2 - 6\rho_2 + 6\rho_1}{3} \tag{27}$$

$$= \frac{U_1^2 h}{8} \cdot \frac{2\rho_2 + 6\rho_1}{3} \tag{28}$$

$$= \frac{U_1^2 h}{8} \cdot \frac{2(\rho_2 + 3\rho_1)}{3}. \tag{29}$$

Step 6 Simplify

Cancel the factor of 2 with the 8 in the denominator:

$$E_{\text{final}} = \frac{U_1^2 h}{4} \cdot \frac{\rho_2 + 3\rho_1}{3}. \tag{30}$$

Distributing, this can be written equivalently as:

$$E_{\text{final}} = \frac{\rho_1 U_1^2 h}{3} + \frac{(\rho_2 - \rho_1)\, U_1^2 h}{12}. \tag{31}$$

$$\boxed{E_{\text{final}} = \frac{\rho_1 U_1^2 h}{3} + \frac{(\rho_2 - \rho_1)\, U_1^2 h}{12}} \tag{32}$$

This matches the expression stated in the original notes.

Comparison and energy loss

From (18) and (32):

$$\Delta E = E_{\text{final}} - E_{\text{initial}}$$

$$= \frac{\rho_1 U_1^2 h}{3} + \frac{(\rho_2 - \rho_1) U_1^2 h}{12} - \frac{\rho_1 U_1^2 h}{2}. \tag{33}$$

Writing everything over 12:

$$
\begin{aligned}
\Delta E &= \frac{U_1^2 h}{12}\left[4\rho_1 + (\rho_2 - \rho_1) - 6\rho_1\right] \\
&= \frac{U_1^2 h}{12}\left[\rho_2 - 3\rho_1 + 4\rho_1 - 6\rho_1 + \rho_2 - \rho_2\right] \\
&= \frac{U_1^2 h}{12}\left[\rho_2 - 3\rho_1\right]. \qquad (34)
\end{aligned}
$$

More cleanly, combining $4\rho_1 + \rho_2 - \rho_1 - 6\rho_1 = \rho_2 - 3\rho_1$:

$$
\Delta E = \frac{U_1^2 h\,(\rho_2 - 3\rho_1)}{12}. \qquad (35)
$$

Since in the physically relevant limit $\rho_2 - \rho_1 \ll \rho_1$, we have $\rho_2 < 3\rho_1$, so $\Delta E < 0$: **kinetic energy decreases after mixing**. The deficit is transferred to potential energy, raising the centre of mass of the dense fluid against gravity.

Summary table.

Quantity	Before mixing	After mixing
Momentum	$\rho_1 U_1 h$	$\frac{U_1 h(\rho_1 + 2\rho_2)}{3}$
Kinetic energy	$\frac{1}{2}\rho_1 U_1^2 h$	$\frac{\rho_1 U_1^2 h}{3} + \frac{(\rho_2 - \rho_1) U_1^2 h}{12}$