The planar laminar jet

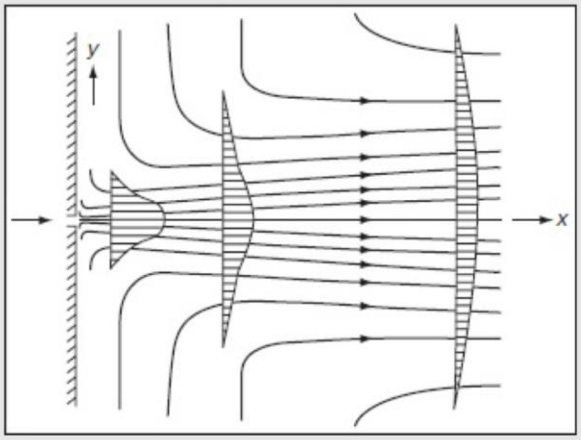

FIGURE 4-23
Definition sketch for a two-dimensional laminar free jet. [*After Schlichting* (*1933a*).]

Since jet spreads at p = constant w/o bounding walls

$$u_x + v_y = 0$$

$$uu_x + vu_y = \nu u_{yy}$$

$$J = \rho \int_{-\infty}^{\infty} u^2 \, dy = \text{constant}$$

$$\psi = \nu^{1/2} x^{1/3} f(\eta) = a(x) f(\eta) \quad \eta = \frac{y}{3\nu^{1/2} x^{2/3}} = \frac{y}{b(x)}$$ scaling assumed

$$\eta_y = b^{-1}, \quad \eta_x = -\frac{y}{b^2} b_x = -\frac{\eta b_x}{b}$$

$$u = \psi_y = af' b^{-1} = \frac{x^{-1/3}}{3} f' \qquad \frac{\nu^{1/2} x^{1/3}}{3\nu^{1/2} x^{2/3}} = \frac{x^{-1/3}}{3}$$

$$v = -\psi_x = -a_x f + af' \frac{\eta b_x}{b}$$

$$v = -\frac{\nu^{1/2}}{3} x^{-2/3} f + \frac{x^{-1/3}}{3}\left(2\nu^{1/2} x^{-1/3}\right)\eta f'$$

$$v = -\frac{1}{3}\nu^{1/2} x^{-2/3}[f - 2\eta f']$$

$$a_x = \frac{\nu^{1/2}}{3} x^{-2/3} \quad b = 3\nu^{1/2} x^{2/3} \quad b_x = \frac{2}{3} 3\nu^{1/2} x^{-1/3} = 2\nu^{1/2} x^{-1/3}$$

$$(a_x b - a b_x) f'^2 - a_x b f f'' - \nu f''' = 0$$

$$a_x b = \frac{\nu^{1/2}}{\cancel{3}} x^{-2/3} \cdot \cancel{3}\nu^{1/2} x^{2/3} = \nu$$

$$a b_x = \nu^{1/2} x^{1/3} \cdot 2\nu^{1/2} x^{-1/3} = 2\nu$$

$$\therefore f''' + \underbrace{f f'' + f'^2}_{\frac{d}{d\eta}(f' f)} = 0$$

BC: $u_y = 0, \quad v = 0, \quad y = 0 \quad u = 0, \quad y = \pm\infty$

$$\therefore f''(0) = 0, \quad f(0) = 0 \quad y = 0$$

$$f'(\pm\infty) = 0$$

$$f'' + f f' = c_1 = 0 \qquad f' = f'' = 0 \quad \eta = \pm\infty$$

let $\beta = \alpha\eta, \quad f = 2\alpha F(\beta)$

$$\beta' = \alpha \qquad f' = 2\alpha F' \alpha = 2\alpha^2 F'$$

$$f'' = 2\alpha^3 F''$$

$$2\alpha^3 F'' + 2\alpha^2 F' \, 2\alpha F = 0$$

$$F'' + 2FF' = 0 \quad \frac{d}{d\eta}(F^2) = 2FF'$$

$$\beta = 0, \quad F = 0, \quad \beta = \pm\infty, \quad F' = 0$$

Also $f'(0) = 1 \qquad u = \frac{x^{-1/3}}{3} f'$

$u(0) = u_{max} f'(0) \quad \therefore f'(0) = 1$ and $F'(0) = 1$

$F' + F^2 = c_2 = 1$ Riccati type

$$\beta = \int_0^F \frac{dF}{1 - F^2} = \frac{1}{2} \ln \left(\frac{1 + F}{1 - F} \right) = \tanh^{-1} F$$

or $F = \tanh \beta = \frac{1 - \exp(-2\beta)}{1 + \exp(-2\beta)}$

$$\frac{dF}{d\beta} = 1 - \tanh^2 \beta$$

$$u = \frac{2}{3} \alpha^2 x^{-1/3} (1 - \tanh^2 \beta) = \frac{2}{3} \alpha^2 x^{-1/3} sech^2 \beta$$

$J = \rho \int_{-\infty}^{\infty} u^2 \; dy = \text{constant}$

$dy = b(x)\, d\eta = 3\nu^{1/2} x^{2/3} d\eta$

$$J = \rho \int_{-\infty}^{\infty} \left(\frac{2\alpha^2}{3x^{1/3}} \text{sech}^2(\alpha\eta) \right)^2 \; 3\nu^{1/2} x^{2/3} \, d\eta$$

$= \frac{16}{9} \rho \nu^{1/2} \alpha^3$

i.e., $\alpha = \left(\frac{9J}{16\sqrt{\rho\mu}} \right)^{1/3} \approx 0.8255 \; \frac{J^{1/3}}{(\rho\mu)^{1/6}}$

$$u_{max} = u(0) = \frac{2\alpha^2}{3} x^{-1/3} = \frac{2}{3} \left(\frac{9J}{16\sqrt{\rho\mu}} \right)^{2/3} \; x^{-1/3}$$

$$= \frac{2}{3} \left(\frac{9}{16} \right)^{2/3} \frac{J^{2/3}}{(\rho\mu x)^{1/3}} \approx 0.4544 \left(\frac{J^2}{\rho\mu x} \right)^{1/3} \propto x^{-1/3}$$

$$u = u_{max}\,\mathrm{sech}^2(\alpha\eta) = u_{max}\,\mathrm{sech}^2\left[\underbrace{0.2752\left(\frac{\rho J}{\mu^2 x^2}\right)^{1/3} y}_{\alpha\eta}\right]$$

note $\mathrm{sech}^2 3 \approx 0.01$

b = width jet = $2y$ distance where $u = 0.01u_{max}$

$$= 2y|_{1\%} \approx 21.8\left(\frac{\mu^2}{\rho J}\right)^{1/3} x^{2/3} \propto x^{2/3}$$

$$\dot{m} = \rho\int_{-\infty}^{\infty} u\, dy = (36\mu\rho J x)^{1/3} \approx 3.302\,(\mu\rho J x)^{1/3} \propto x^{1/3}$$

Jet drags ambient fluid along its path as it develops. Solution not valid near $x = 0$, as BL approximation not valid since $Re = \frac{\dot{m}}{\mu} \sim \left(\frac{\rho J x}{\mu^2}\right)^{1/3}$ per unit depth not large. Profile S shape with inflection points ∴ unstable transition turbulence $Re_{crit} \sim 30$.