Planar jet

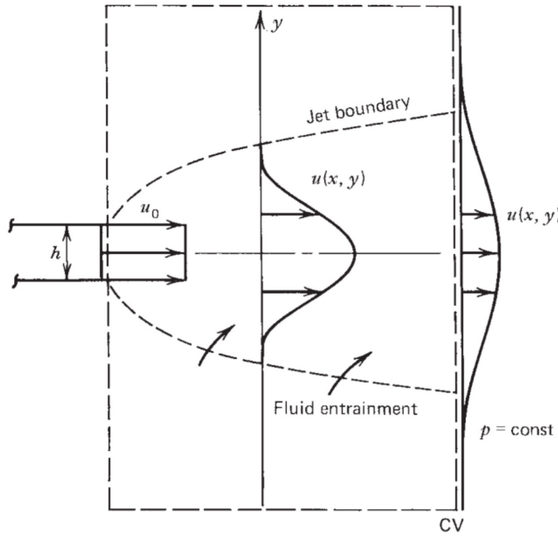

Figure 20.21 Plane laminar jet into an infinite medium.

width h

$Re = \frac{u_0 h}{\nu}$ large

At sufficient distance orifice all jets decay with similarity independent u_0

Assume $p_{inlet} = p_{exit}$

i.e., p = constant and $p_x = 0$ and $p_y = 0$ along with BL assumptions

$$\int_{cs} \rho \underline{V} \cdot \underline{n}\, dA = 0 = -\rho u_0^2 h + \int_{-\infty}^{\infty} \rho u \; dy + \dot{m}_t + \dot{m}_s$$

$$\dot{m}_t + \dot{m}_s = \rho u_0 h - \int_{-\infty}^{\infty} \rho u \; dy$$

$$\int_{cs} \rho u \; \underline{V} \cdot \underline{n}\, dA = 0 = -\rho u_0^2 h + \int_{-\infty}^{\infty} \rho u^2 \; dy + u_{t,b}(\dot{m}_t + \dot{m}_s)$$

$u_{t,b} = 0$

$M = \rho u_0^2 h = \int_{-\infty}^{\infty} \rho u^2 \; dy = \text{constant}$ M = only property flow at orifice carried downstream

Since jet entrains ambient fluid, flow rate $f(x)$

Assume $\psi(x, y)$ exists with $u = \psi_y, \quad v = -\psi_x$

$$u u_x + v u_y = \nu u_{yy}$$

$$\psi_y \psi_{yx} - \psi_x \psi_{yy} = \nu \psi_{yyy}$$

Ax^p scales ψ and Bx^q scales y In this case p and q not assumed.

$$\psi = Ax^p f(\eta) = a(x) f(\eta) \qquad \eta = \frac{y}{Bx^q} = \frac{y}{b(x)}$$

p, q part of solution and A, B make f and η dimensionless

$$(a_x b - a b_x) f'^2 - a_x b f f'' - \nu f''' = 0$$

$$a_x = Apx^{p-1} \qquad a_x b = Apx^{p-1}\, Bx^q = ABpx^{p+q-1}$$

$$b_x = Bqx^{q-1} \qquad a b_x = Ax^p\, Bqx^{q-1} = ABqx^{p+q-1}$$

$$f'^2 (ABpx^{p+q-1} - ABqx^{p+q-1}) - ABpx^{p+q-1} f f'' - \nu f''' = 0$$

$$\frac{ABx^{p+q-1}}{\nu} [(p-q) f'^2 - p f f''] = f'''$$

$$u = 0 \quad \eta = \pm\infty \qquad u_y = 0 \quad \eta = 0 \qquad v = 0 \quad \eta = 0$$

$$f'(\pm\infty) = 0 \qquad f'(0) = 0 \qquad f(0) = 0$$

For similarity $\frac{ABx^{p+q-1}}{\nu} \neq f(x) \Rightarrow p + q = 1$

$$M = \text{constant} = \int_{-\infty}^{\infty} \rho u^2 \, dy = \int_{-\infty}^{\infty} \rho \psi_y^2 \, dy$$

$$\psi_y = \frac{Ax^p f'}{Bx^q} = \frac{Ax^p}{Bx^q} f'(\eta) \qquad \eta = \frac{y}{Bx^q}$$

$$M = \int_{-\infty}^{\infty} \rho \frac{A^2 x^{2p}}{(Bx^q)^2} Bx^q f'^2 \, d\eta \qquad Bx^q d\eta = dy$$

$$= \rho A^2 B^{-1} x^{2p-q} \int_{-\infty}^{\infty} f'^2 \, d\eta \neq f(x) \Rightarrow 2p - q = 0 \quad 2p = q$$

$$p + 2p = 1 \Rightarrow p = \frac{1}{3} \qquad q = \frac{2}{3}$$

$$-f''' + \frac{AB}{\nu}\left[-\frac{1}{3}f'^2 - \frac{1}{3}ff''\right] = 0$$

$$f''' + \frac{AB}{3\nu}\underbrace{[f'^2 + ff'']}_{\frac{d}{d\eta}(ff')} = 0$$

integrate $\eta = \pm\infty \quad f'' + \frac{AB}{3\nu}f'f = C_1 = 0 \qquad \eta = \pm\infty \quad f' = f'' = 0$

$$\frac{d}{d\eta}(f') + \frac{AB}{3\nu}\frac{d}{d\eta}\left(\frac{f^2}{2}\right) = 0$$

$$f' + \frac{AB}{6\nu}f^2 = C_2$$

$$\eta = 0, f(0) = 0, C_2 = f'(0)$$

$$u = \frac{Ax^{1/3}}{Bx^{2/3}} = \frac{A}{B}x^{-1/3}f'$$

$$u(0) = u_{max}f'(0) \qquad f'(0) = 1 \quad C_2 = 1 \qquad a = \frac{AB}{6\nu}$$

$$\frac{df}{d\eta} + af^2 = 1 \qquad df = (1 - af^2)\,d\eta \qquad \frac{df}{1 - af^2} = d\eta$$

Insert #1 Jet 2D-1. Let a = 1

$$f = \tanh\eta \quad f' = \text{sech}^2\eta$$

$A = \left(\frac{9\nu M}{2\rho}\right)^{1/3} \qquad B = \left(\frac{48\nu^2\rho}{M}\right)^{1/3}$ Insert #1

$$AB = \left(\frac{9\nu \cancel{M}}{2\cancel{\rho}}\cdot\frac{48\nu^2\cancel{\rho}}{\cancel{M}}\right)^{1/3} = (216\nu^3)^{1/3} = 6\nu \;\Rightarrow a = 1$$

$$u = \frac{A}{B}x^{-1/3}f'(\eta) = u_{max}\,\text{sech}^2\eta$$

$$\frac{3\times 3}{2\times 3\times 16}=\frac{3}{32} \qquad AB^{-1}=\left(\frac{9\nu M}{2\rho}\frac{M}{48\nu^2\rho}\right)^{1/3}=\left(\frac{3}{32}\frac{M^2}{\nu\rho^2}\right)^{1/3}$$

$$u=\left(\frac{3}{32}\frac{M^2}{\nu\rho^2}\right)^{1/3} x^{-1/3} f'(\eta)=u_{max}\,\mathrm{sech}^2\eta$$

$u_{max} \propto x^{-1/3}$

$\eta=\frac{y}{b(x)}=y\left(\frac{M}{48\nu^2\rho}\right)^{1/3} x^{-2/3}$ Insert #2

Jet thickness $\frac{u}{u_{max}}$ = fraction = locus point $y\,x^{-2/3}=$ constant

$h(x)=\mathrm{C}\left(\frac{48\nu^2\rho}{M}\right)^{1/3} x^{2/3} \propto x^{2/3}$ Insert #3

Viscous forces at jet edge accelerate ambient fluid and entrain fluid at rate:

$Q=\int_{-\infty}^{\infty} u\,dy=\left(\frac{36M\nu}{\rho}\right)^{1/3} x^{1/3} \propto x^{1/3}$ Insert #4

$x=0$ flow singularity ($u_{max}=\infty, h=Q=0$) due breakdown BL assumptions, i.e., need replace $x=x-x_0$ where x_0 = effective origin.

Insert #4

$$Q=\int_{-\infty}^{\infty} u\,dy=\int_{-\infty}^{\infty} u_{max}\,\mathrm{sech}^2\eta\,dy \qquad \eta=\frac{y}{Bx^{2/3}}$$

$$=u_{max}Bx^{2/3}\underbrace{\int_{-\infty}^{\infty}\mathrm{sech}^2\eta\,d\eta} \qquad d\eta=\frac{dy}{Bx^{2/3}}$$

$$=\tanh\eta\Big|_{-\infty}^{\infty}=2$$

$$=\frac{A}{B}x^{-1/3}Bx^{2/3}$$

$$=\left[\frac{9\nu M}{2\rho}\right]^{1/3} 2x^{1/3}$$

$$=\left[\frac{36M\nu}{\rho}\right]^{1/3} x^{1/3}$$

Insert # 1

$a = \frac{AB}{6\nu} = 1$

$$M = \frac{\rho A^2}{B} \underbrace{\int_{-\infty}^{\infty} f'^2 \, d\eta}_{4/3} \qquad f' = \operatorname{sech}^2 \eta$$

Insert # 2 Jet 2D-1

$$B = \frac{4}{3} \frac{\rho A^2}{M} = \frac{6\nu}{A}$$

$$A^3 = \frac{3M6\nu}{4\rho} \qquad A = \left[\frac{9M\nu}{2\rho}\right]^{1/3} \qquad B = \left[\frac{48\nu^2\rho}{M}\right]^{1/3}$$

Insert #2

$$\eta = y/g(x) = \frac{y}{Bx^{2/3}} = y\left[\frac{M}{48\nu^2\rho}\right]^{1/3} x^{-2/3}$$

Insert #3

jet thickness u/u_{max} = fraction

= locus of points $yx^{-2/3}$

$\operatorname{sech}^2 \eta$ = constant = constant

$$\frac{h(x)}{B} x^{-2/3} = c \qquad h(x) \propto x^{2/3}$$

$$h(x) = cBx^{2/3}$$

$$= c\left[\frac{48\nu^2\rho}{M}\right]^{1/3} x^{2/3}$$