Plane laminar jet

($v \neq 0$ due to entrainment)

Entrainment = jet entrains surrounding fluid due to viscous vorticity diffusion at the outer edge of the jet

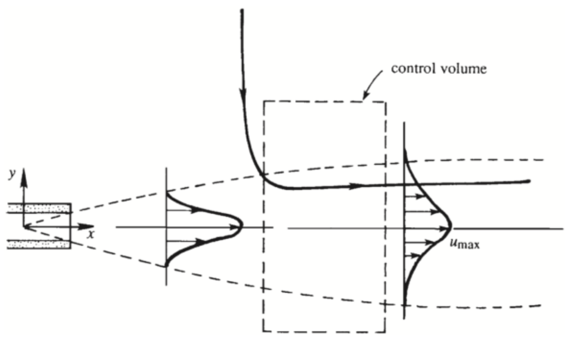


FIGURE 9.27 Simple laminar two-dimensional free jet. A narrow slot injects fluid horizontally with an initial momentum flux J into a nominally quiescent reservoir of the same fluid. The region of horizontally moving fluid slows and expands as x increases. A typical streamline showing entrainment of surrounding fluid is indicated.

Near field depends on IC, whereas far field achieves similarity due to absence externally imposed length scale in downstream direction.

BL assumptions: $Re = \frac{u_0 x}{\nu} \gg 1$ u_0 = uniform inlet velocity ρ, μ constant

$\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$, $v \ll u$, $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$ note: $Re_{crit} = 4$ $Re_{trans} \sim 30$

(1) $u_x + v_y = 0$

(2) $uu_x + vu_y = \nu u_{yy}$

$$\left.\begin{array}{l} 1\ u(y = \pm\infty) = 0 \\ 2\ v(y = 0) \end{array}\right\} x > 0$$

$$3\ \ u(x = x_0) = \tilde{u}(\mathrm{y}) \qquad -\frac{h}{2} \leq y \leq \frac{h}{2}$$

$$4\ \ u_y(y = 0) = 0$$

1 = Infinity condition, but $v \neq 0$

2 and 4 Symmetry conditions

3 Inlet condition

$\int_{-\infty}^{\infty}[u\cdot(1)+\mathrm{LHS}(2)]\ dy = \int_{-\infty}^{\infty} 2u\,u_x\,dy + \int_{-\infty}^{\infty}(uv_y+vu_y)\ dy =$
$\int_{-\infty}^{\infty}(\nu u_{yy})dy = \nu u_y|_{-\infty}^{\infty} = \frac{\tau}{\rho}|_{-\infty}^{\infty} \qquad \tau = \mu u_y$

$$\frac{d}{dx}\int_{-\infty}^{\infty} u^2\ dy + 2uv|_{-\infty}^{\infty} = \nu u_y|_{-\infty}^{\infty}$$

$u(y\ =\ \pm\infty) = 0$ therefore $u_y(y\ =\ \pm\infty) = 0$

$$\frac{d}{dx}\int_{-\infty}^{\infty}\rho u^2\ dy = 0$$

Streamwise momentum flux preserved, i.e., zero drag (zero drag and free stream velocity version CV flat plate conservation momentum). Similar derivation momentum integral equation #3 except integral limits $\pm\infty$

$\therefore \int_{-\infty}^{\infty} u^2\ dy = \text{constant} = \int_{-\infty}^{\infty}\tilde{u}^2(y)\ dy = \frac{J}{\rho}$ key feature #1

J = momentum flux per unit span = constant

Where $u = \tilde{u}(y)$ at $x = x_0$ $\qquad \tilde{u}(y)$ = known initial velocity profile

Similarity solution far enough downstream x_0 such that BL equations valid and w/o influence $\tilde{u}(y)$. Since BL equations same Blasius but different BC, seek solution of same form. Note scaling $u_0(x)$ not yet determined.

$$\psi = u_0(x)\,\delta(x)\,f(\eta) \quad \eta = \frac{y}{\delta(x)} \quad \delta(x) = \left[\frac{\nu x}{u_0}\right]^{1/2}$$

$u_0(x) = u(x,0) = u_{max}$

$u = \psi_y = u_0\,\delta\,f'\,\eta_y = u_0 f' \qquad u(0) = u_0 \Rightarrow f'(0) = 1$

$$\frac{J}{\rho} = \int_{-\infty}^{\infty} u^2 \, dy = u_0^2 \int_{-\infty}^{\infty} f'^2(\eta)\, d\eta = u_0^2 \delta \int_{-\infty}^{\infty} f'^2(\eta)\, d\eta \; = u_0^2 \delta C$$

$$= u_0^2 \left[\frac{\nu x}{u_0}\right]^{1/2} C$$

$d\eta = \; dy/\delta \quad C \; =$ non dimensional constant

$$\frac{J}{\rho} = C\, u_0^{3/2} (\nu x)^{1/2}$$

$$\frac{J^2}{\rho^2} = C^2 u_0^3 \nu x \qquad u_0^3 = \frac{J^2}{C^2 \rho^2 \nu x}$$

$$u_0(x) = \left[\frac{J^2}{C^2 \rho^2 \nu x}\right]^{1/3} \qquad \delta(x) = \left[\frac{\nu x}{u_0}\right]^{1/2} = \left[\frac{C \rho \nu^2 x^2}{J}\right]^{1/3}$$

$$(u_0 \delta)^3 = \frac{J^2}{\rho^2 C^2 \nu x} \times \frac{C \rho \nu^2 x^2}{J} = \frac{J \nu x}{\rho C}$$

$$\psi = \underbrace{\left[\frac{J \nu x}{\rho C}\right]^{1/3}}_{a(x)} f(\eta) \qquad \eta = \frac{y}{\underbrace{\left[\frac{C \rho \nu^2 x^2}{J}\right]^{1/3}}_{b(x)}}$$

$$\psi_y \psi_{yx} - \psi_x \psi_{yy} = \nu \psi_{yyy}$$

$$\psi = a(x) f(\eta) \qquad \eta = \frac{y}{b(x)} = y b^{-1} \qquad \eta_y = b^{-1} \quad \eta_{yy} = 0$$

$$\eta_x = -\frac{y}{b^2} b_x = -\eta \frac{b_x}{b}$$

$$u = \psi_y = af'b^{-1} = \frac{a}{b}f'$$

$$u_x = (ab^{-1})_x f' + (ab^{-1})f''\left(-\frac{\eta b_x}{b}\right)$$

$$u_y = (ab^{-1})f''b^{-1} = (ab^{-2})f''$$

$$u_{yy} = (ab^{-3})f'''$$

$$v = -\psi_x = -\left[a_x f + af'\left(-\frac{\eta b_x}{b}\right)\right] = -a_x f + f'\left(\frac{\eta a b_x}{b}\right)$$

$$(ab^{-1})f'\left[(ab^{-1})_x f' + (ab^{-1})f''\left(-\frac{\eta b_x}{b}\right)\right] \\ + \left[-a_x f + f'\left(\frac{\eta a b_x}{b}\right)\right](ab^{-2})f'' = \nu(ab^{-3})f'''$$

$$(ab^{-1})(ab^{-1})_x f'^2 - (ab^{-2})\frac{\eta b_x}{b}\, ff'' - a_x(ab^{-2})ff'' + \\ (ab^{-2})\frac{\eta a b_x}{b}\, ff'' = \nu(ab^{-3})f'''$$

$$(ab^{-1})(ab^{-1})_x f'^2 - (ab^{-2})\frac{\eta b_x}{b}\, ff'' - \nu(ab^{-3})f''' = 0$$

$$\frac{\cancel{a}}{b}\left(\frac{a_x}{b} - \frac{a}{b^2}b_x\right)f'^2 - \frac{a_x\cancel{a}}{b^2}ff'' - \nu\frac{\cancel{a}}{b^3}f''' = 0$$

$$b^2\left(\frac{a_x}{b} - \frac{a}{b^2}b_x\right)f'^2 - a_x bff'' - \nu f''' = 0$$

$$(a_x b - ab_x)f'^2 - a_x bff'' - \nu f''' = 0$$

$$a = \left[\frac{J\nu x}{\rho C}\right]^{1/3} = \left(\frac{J\nu}{\rho C}\right)^{1/3} x^{1/3} \qquad a_x = \left(\frac{J\nu}{\rho C}\right)^{1/3}\frac{1}{3}x^{-2/3}$$

$$b = \left[\frac{C\rho\nu^2 x^2}{J}\right]^{1/3} = \left(\frac{C\rho\nu}{J}\right)^{1/3} x^{2/3} \qquad b_x = \left(\frac{C\rho\nu}{J}\right)^{1/3}\frac{2}{3}x^{-1/3}$$

$$a_x b = \left(\frac{J\nu}{\rho C}\right)^{1/3} \frac{1}{3} x^{-2/3} \left(\frac{C\rho\nu^2}{J}\right)^{1/3} x^{2/3} = \left(\frac{J\nu}{\rho C}\frac{C\rho\nu^2}{J}\right)^{1/3} \frac{1}{3} = \frac{\nu}{3}$$

$$ab_x = \left(\frac{J\nu}{\rho C}\right)^{1/3} x^{1/3} \left(\frac{C\rho\nu^2}{J}\right)^{1/3} \frac{2}{3} x^{-1/3} = \left(\frac{\cancel{J}\nu}{\cancel{\rho C}}\frac{\cancel{\rho C}\nu^2}{\cancel{J}}\right)^{1/3} \frac{2}{3} = \frac{2\nu}{3}$$

$$\left(\frac{\nu}{3} - \frac{2\nu}{3}\right) f'^2 - \frac{\nu}{3} - \nu f''' = 0$$

i.e., $3f''' + ff'' + f'^2 = 0 \quad (1)$

$$3\frac{d}{d\eta}(f'') + \frac{d}{d\eta}(f'f) = 0$$

BC $f' = 0 \quad \eta \to \pm\infty \quad f' = 1, \quad f = 0 \quad \eta = 0$

$u(x, \pm\infty) = 0 \qquad u(x, 0) = u_0(x) \qquad v(x, 0) = 0$

integrating (1): $3f'' + ff' = c_1$ (2)

at $\eta = \pm\infty, \quad f' = 0 \therefore f'' = 0 \therefore c_1 = 0$

integrating (2): $3f' + \frac{f^2}{2} = c_2$ (3)

at $\eta = 0$ $f' = 1, f = 0 \Rightarrow c_2 = 3$

$$3\frac{df'}{d\eta} + \frac{d(\frac{f^2}{2})}{d\eta} = 0$$

$$3\frac{df}{d\eta} + \frac{f^2}{2} = 3\frac{df}{d\eta} + \frac{f^2}{6} = 1 \quad df = \left(1 - \frac{f^2}{6}\right) d\eta \quad \Rightarrow \int \frac{df}{1 - \frac{f^2}{6}} = \int d\eta$$

Insert #1 $f = \sqrt{6}\tanh\left(\frac{\eta}{\sqrt{6}}\right) + c_3$

$f(0) = 0 \Rightarrow c_3 = 0 \quad f' = \mathrm{sech}^2\left(\frac{\eta}{\sqrt{6}}\right)$

$$u(x,y) = u_0(x) f'(\eta) = \left(\frac{J^2}{C^2\rho^2\nu x}\right)^{1/3} \operatorname{sech}^2\left(\frac{y}{\sqrt{6}}\left[\frac{J}{C\rho\nu^2 x^2}\right]^{1/3}\right)$$

$C = \int_{-\infty}^{\infty} f'^2(\eta)\, d\eta = \int_{-\infty}^{\infty} \operatorname{sech}^4\left(\frac{\eta}{\sqrt{6}}\right) d\eta = \frac{4\sqrt{6}}{3}$ Insert #2

$$\dot{m} = \int_{-\infty}^{\infty} \rho u_0(x) f'(\eta)\, dy = \rho u_0(x)\delta(x) \int_{-\infty}^{\infty} f'(\eta)\, d\eta =$$

$\rho u_0(x)\delta(x)\, f|_{-\infty}^{\infty} = \rho u_0(x)\delta(x)\, 2\sqrt{6}$ Insert #3

$$u_0(x) = \left[\frac{J^2}{C^2\rho^2\nu x}\right]^{1/3} = \left[\frac{3J^2}{32\rho^2\nu x}\right]^{1/3} \propto x^{-1/3}$$

$$\delta(x) = \left[\frac{C\rho\nu^2 x^2}{J}\right]^{1/3} = \left[\frac{4\sqrt{6}J\rho\nu^2 x}{3J}\right]^{1/3} \propto x^{2/3}$$

$$u_0\delta = \left[\frac{J^2}{C^2\rho^2\nu x}\frac{C\rho\nu^2 x^2}{J}\right]^{1/3} = \left[\frac{J\nu x}{C\rho}\right]^{1/3} = \left[\frac{3J\nu x}{4\sqrt{6}\rho}\right]^{1/3}$$

$\dot{m} = (36J\,\rho^2\nu x)^{1/3} \propto x^{1/3}$ increases downstream due entrainment via ν

key feature #2

$$\begin{aligned} v = -\psi_x &= -a_x f + f'\left(\eta\frac{ab_x}{b}\right) \\ &= -\frac{1}{3}\left(\frac{J\nu}{C\rho x^2}\right)^{1/3} f + f'\left(\eta\frac{2\nu}{3}\left[\frac{J}{C\rho\nu x^2}\right]^{1/3}\right) \\ &= -\frac{1}{3}\left(\frac{J\nu}{C\rho x^2}\right)^{1/3} [f - 2\eta f'] \end{aligned}$$

or $\frac{v}{u_0(x)} = -\frac{[f-2\eta f']}{3\sqrt{Re_x}}$, $\qquad Re_x = \frac{x u_0(x)}{\nu}$

$$f(\eta) = \pm\sqrt{6}, \qquad 2\eta f'(\eta) = 2\eta \operatorname{sech}^2\left(\frac{\eta}{\sqrt{6}}\right) = 0, \qquad \eta = \pm\infty$$

$$\therefore \frac{v}{u_0(x)} = \mp \frac{\sqrt{6}}{3\sqrt{Re_x}}, \qquad \eta = \pm\infty \text{ entrainment due } v \text{ from above/below}$$

The jet spreads as x ↑. Using similar idea BL δ at u/U 99% define half width jet h99 as y location where u falls to 1% of $y = 0$ (i.e., $0.01\, u_0(x)$)

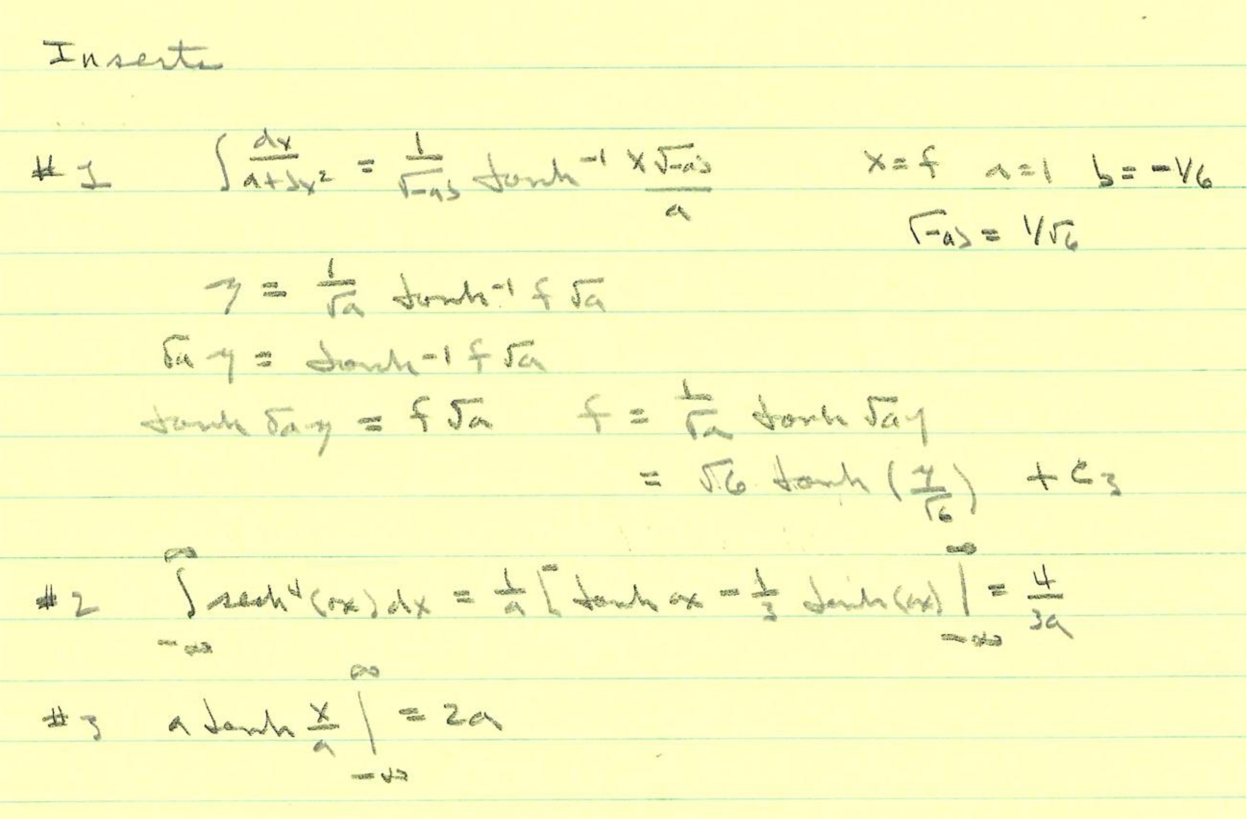
$$\operatorname{sech}^2\left(\frac{h_{99}}{\sqrt{6}}\left[\frac{J}{C\rho\nu^2x^2}\right]^{1/3}\right) = 0.01 \;\Rightarrow\; \frac{h_{99}}{\sqrt{6}}\left[\frac{J}{C\rho\nu^2x^2}\right]^{1/3} = 2.2924$$

$$h_{99} = 5.6152\left[\frac{C\rho\nu^2x^2}{J}\right]^{1/3} = 5.6152\left[\frac{4\sqrt{6}}{3}\frac{\rho\nu^2x^2}{J}\right]^{1/3}$$

jet width ∝ $x^{2/3}$ and ν increases width, whereas J decreases width

$$Re_x = \frac{xu_0(x)}{\nu} = \left(\frac{3Jx}{4\sqrt{6}C\rho\nu^2}\right)^{1/3} \qquad Re_{h_{99}} = 5.6152\left(\frac{3Jx}{4\sqrt{6}\rho\nu^2}\right)^{1/3}$$

Unstable: $Re_x \gg 1$ due inflection points



Inserts

#1 $\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{-ab}} \tanh^{-1} \frac{x\sqrt{-ab}}{a}$ $\qquad x = f \quad a = 1 \quad b = -1/6$

$\sqrt{-ab} = 1/\sqrt{6}$

$\eta = \frac{1}{\sqrt{a}} \tanh^{-1} f\sqrt{a}$

$\sqrt{a}\,\eta = \tanh^{-1} f\sqrt{a}$

$\tanh \sqrt{a}\,\eta = f\sqrt{a} \qquad f = \frac{1}{\sqrt{a}} \tanh \sqrt{a}\,\eta$

$= \sqrt{6} \tanh\left(\frac{\eta}{\sqrt{6}}\right) + c_3$

#2 $\int_{-\infty}^{\infty} \operatorname{sech}^4(ax)\,dx = \frac{1}{a}\left[\tanh ax - \frac{1}{3}\tanh(ax)\right|_{-\infty}^{\infty} = \frac{4}{3a}$

#3 $a \tanh \frac{x}{a}\Big|_{-\infty}^{\infty} = 2a$