

Axisymmetric round/circular jet

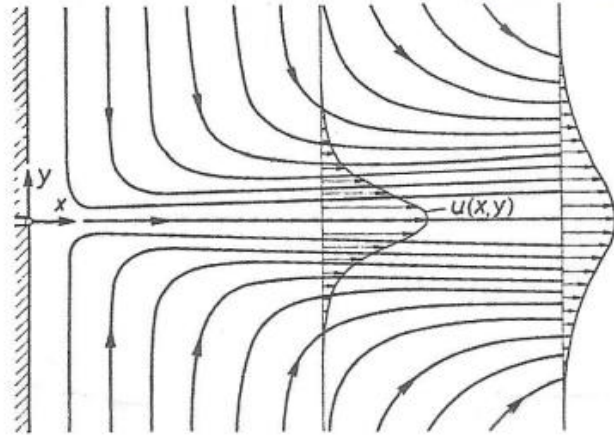


Fig. 11.5. Streamline pattern for a circular laminar jet

$$Re = \frac{u_0 r_0}{\nu} \text{ large } \frac{\partial}{\partial r} \gg \frac{\partial}{\partial x} \quad (x, r) \text{ coordinates and } u = u(x, r)$$

$p = \text{constant}$ and jet narrow with slow growth rate

$$\text{As with 2D jet: } J = 2\pi\rho \int_0^\infty u^2 r \, dr = \text{constant} \quad (1)$$

$$u_x + \frac{1}{r}(rv)_r = 0 \quad (2) \quad u_x + v_r + \frac{v}{r} = 0$$

$$uu_x + vu_r = \frac{\nu}{r}(ru_r)_r \quad (3) \quad \text{RHS} = \frac{\nu}{r}(u_r + ru_{rr}) = \nu \left(\frac{u_r}{r} + u_{rr} \right)$$

$$u_y = 0 \quad (u = u_{max}), \quad v = 0, \quad r = 0$$

$$u = 0, \quad r = \infty$$

Assume similarity solution for $u(x, y)$ with jet width $\propto x^n$ and stream function

$$\psi \propto x^p F(\eta) \text{ and } \eta = \frac{r}{x^n} = rx^{-n} \quad r = \eta x^n \propto x^n$$

p and n determined, $J \neq f(x)$ and LHS & RHS(3) same order of magnitude

$$\text{Note: } u = \frac{1}{r}\psi_r \text{ and } v = -\frac{1}{r}\psi_x \quad m^{-1} \times \psi_r = \frac{m}{s}$$

$$u \propto x^{p-2n}, \text{ i.e., } r^{-1}\psi_r \propto \frac{x^p}{x^n} \cdot x^{-n} = x^{p-2n}$$

$$u_x \propto x^{p-2n-1} \quad u_r \propto x^{p-2n} \cdot x^{-n} = x^{p-3n}$$

$$u_{rr} \propto x^{p-3n} \cdot x^{-n} \propto x^{p-4n}$$

$$\text{From (1) } u^2 r dr \propto 2p - 4n + 2n = 0 \quad p = n$$

$$\text{From (3) } uu_x \propto r^{-1}(ru_r)_r \quad 2p - 4n - 1 = p - 4n$$

$$-2p - 1 = -3p \quad p = 1$$

$$\therefore \psi = vx F(\eta) \quad \eta = \frac{r}{x}, \quad r = x\eta \quad \eta_r = x^{-1}, \quad \eta_{rr} = 0$$

$$\eta_x = -rx^{-2} = -\frac{\eta}{x}$$

$$u = r^{-1}\psi_r = r^{-1}vx F' x^{-1} = \frac{vF'}{r} = \frac{vF'}{x\eta}$$

$$\begin{aligned} v &= -r^{-1}\psi_x = -r^{-1}v[F - xF' \eta/x] = -\frac{v}{r}[F - \eta F'] = v \left[\frac{\eta}{r} F' - \frac{F}{r} \right] \\ &= \frac{v}{x} [F' - F/\eta] \end{aligned}$$

$$u_x = vr^{-1}F''(-rx^{-2}) = -vF''x^{-2}$$

$$u_r = v \left[-\frac{F'}{r^2} + \frac{1}{r} \frac{F''}{x} \right] = \frac{v}{r} \left[-\frac{F'}{r} + \frac{F''}{x} \right] = v \left[-\frac{F'}{x^2\eta^2} + \frac{F''}{x^2\eta} \right]$$

$$\begin{aligned} u_{rr} &= v \left[-r^{-2} \left(-\frac{F'}{r} + \frac{F''}{x} \right) + r^{-1} \left(\frac{F'}{r^2} - \frac{F''}{rx} + \frac{F'''}{x^2} \right) \right] = \\ &= v \left[\frac{2F'}{r^3} - \frac{2F''}{xr^2} + \frac{F'''}{x^2r} \right] = v \left[\frac{2F'}{x^3\eta^3} - \frac{2F''}{x^3\eta^2} + \frac{F'''}{x^3\eta} \right] \end{aligned}$$

$$\begin{aligned}
& \frac{vF'}{x\eta} \left[-v \frac{F''}{x^2} \right] + \frac{v}{x} \left[F' - \frac{F}{\eta} \right] \left[v \left(-\frac{1}{x^2\eta^2} + \frac{F''}{x^2\eta} \right) \right] \\
&= v \left[v \left(-\frac{F'}{x^3\eta^3} + \frac{F''}{x^3\eta^2} \right) + v \left(\frac{2F'}{x^3\eta^3} - \frac{2F''}{x^3\eta^2} + \frac{F'''}{x^3\eta} \right) \right] \\
& F'F'' \left[-\frac{v^2}{x^3\eta} + -\frac{v^2}{x^3\eta} \right] + F'^2 \left[-\frac{v^2}{x^3\eta^2} \right] + FF' \left[\frac{v^2}{x^3\eta^3} \right] - FF'' \left[\frac{v^2}{x^3\eta^2} \right] \\
&= -F' \left[\frac{v^2}{x^3\eta^3} \right] + F'' \left[\frac{v^2}{x^3\eta^2} \right] + F' \left[\frac{2v^2}{x^3\eta^3} \right] - F'' \left[\frac{2v^2}{x^3\eta^2} \right] \\
&+ F''' \left[\frac{v^2}{x^3\eta} \right]
\end{aligned}$$

$$-\frac{F'^2}{\eta^2} + \frac{FF'}{\eta^3} - \frac{FF''}{\eta^2} = -\frac{F'}{\eta^3} + \frac{F''}{\eta^2} + \frac{2F'}{\eta^3} - \frac{2F''}{\eta^2} + \frac{F'''}{\eta}$$

$$\frac{FF'}{\eta^2} - \frac{F'^2}{\eta} - \frac{FF''}{\eta} = \frac{F'}{\eta^2} - \frac{F''}{\eta} + F''' = \frac{d}{d\eta} \left(F'' - \frac{F'}{\eta} \right)$$

$$\frac{d}{d\eta} \left(-\frac{FF'}{\eta} \right) = \eta^{-2} FF' - \eta^{-1} (F'^2 + FF'')$$

$$\frac{d}{d\eta} \left(-\frac{F'}{\eta} \right) = -\frac{F''}{\eta} + \frac{F'}{\eta^2}$$

$$u = u_{max} \text{ and } v = 0 \quad y = 0$$

$$F'(0) = 0, \quad F(0) = 0$$

Since u even $f(\eta)$, $\frac{F'}{\eta} = \text{even}$, $F' = \text{odd}$, $F = \text{even}$

$$1^{\text{st}} \text{ integration } \frac{d}{d\eta} \left(-\frac{FF'}{\eta} \right) = \frac{d}{d\eta} \left(F'' - \frac{F'}{\eta} \right)$$

$$FF' = F' - \eta F'' + C_1 \quad F(0) = F'(0) = 0 \therefore C_1 = 0$$

rewrite equation: $\frac{d}{d\eta} \left(\frac{F^2}{2} \right) = 2F' - \frac{d}{d\eta} (\eta F')$

$$\frac{F^2}{2} = 2F - \eta F' + C_2 \quad C_2 = 0$$

same reasoning C_1

$$\text{i.e., } \left(2F - \frac{F^2}{2} \right)^{-1} F' = \eta^{-1}$$

$$\frac{1}{2} \ln \left(\frac{F}{4 - F} \right) = \ln \eta + C$$

$$\ln \left(\frac{F}{4 - F} \right) = 2 \ln \eta + 2C = \ln \eta^2 + 2C = \ln \eta^2 + \ln a = \ln(a\eta^2)$$

$$\frac{F}{4 - F} = a\eta^2$$

$$F = a\eta^2(4 - F) = 4a\eta^2 - a\eta^2 F$$

$$F = \frac{4a\eta^2}{1 + a\eta^2}$$

$$2\sqrt{a}\eta = \xi \quad \xi^2 = 4a\eta^2 \quad \frac{\xi^2}{4} = a\eta^2$$

$\gamma = 2\sqrt{a} = \text{constant of integration}$

$$\text{i.e., } \xi = \gamma\eta = \frac{\gamma r}{x} \quad d\xi = \gamma d\eta \quad \xi_\eta = \gamma$$

$$F = \frac{\xi^2}{1 + \frac{1}{4}\xi^2} = \xi^2 \left(1 + \frac{1}{4}\xi^2 \right)^{-1}$$

$$\int \frac{dF}{\frac{1}{2}(4F - F^2)}$$

$$dF = dx$$

$$X = a + bx + cx^2$$

$$a = 0, \quad b = 1, \quad c = -\frac{1}{2}$$

$$q = 4ac - b^2 = -4 \quad \sqrt{-q} = 2$$

$$\frac{1}{2} \ln \left(\frac{-F}{-F + 4} \right) = \frac{1}{2} \ln \left(\frac{F}{4 - F} \right)$$

$$\begin{aligned}\frac{dF}{d\xi} &= \left[2\xi \left(1 + \frac{1}{4}\xi^2\right)^{-1} - \xi^2 \left(1 + \frac{\xi^2}{4}\right)^{-2} \left(\frac{\xi}{2}\right) \right] \\ &= \left[2\xi \left(1 + \frac{1}{4}\xi^2\right)^{-1} - \frac{\xi^3}{2} \left(1 + \frac{\xi^2}{4}\right)^{-2} \right]\end{aligned}$$

$$= \frac{2\xi \left(1 + \frac{\xi^2}{4}\right) - \frac{\xi^3}{2}}{\left(1 + \frac{\xi^2}{4}\right)^2} = 2\xi + \frac{\xi^3}{2} - \frac{\xi^3}{2}$$

$$u = \frac{v F'}{x \eta} = \frac{v}{x \eta} \gamma \frac{dF}{d\xi} = \frac{v}{x \xi} \gamma^2 \frac{dF}{d\xi}$$

$$u = \frac{v}{x} \frac{2\gamma^2}{\left(1 + \frac{\xi^2}{4}\right)^2}$$

$$v = \frac{v}{x} \left(F' - \frac{F}{\eta}\right) = \frac{v}{x} \gamma \left(\frac{dF}{d\xi} - \frac{F}{\xi}\right) = \frac{v}{x} \gamma \left[\frac{2\xi}{\left(1 + \frac{\xi^2}{4}\right)^2} - \frac{\xi}{\left(1 + \frac{\xi^2}{4}\right)} \right]$$

$$v = \frac{v}{x} \gamma \left[\frac{\xi - \frac{\xi^3}{4}}{\left(1 + \frac{\xi^2}{4}\right)^2} \right]$$

Determine γ from J

$$J = 2\pi\rho \int_0^\infty u^2 r \, dr$$

$$= 2\pi\rho \int_0^\infty \left[\frac{v}{x} \frac{2\gamma^2}{\left(1 + \frac{\xi^2}{4}\right)^2} \right]^2 r \, dr$$

$$= 2\pi\rho \frac{v^2}{x^2} 4\gamma^4 \frac{x^2}{\gamma^2} \int_0^\infty \frac{\xi d\xi}{\left(1 + \frac{\xi^2}{4}\right)^4} = 8\pi\rho v^2 \gamma^2 \left[\frac{-2}{3\left(1 + \frac{\xi^2}{4}\right)^3} \right]_0^\infty$$

$$\xi = \frac{\delta r}{x}, \quad r = \frac{x}{\delta}$$

$$x = a + bx + cx^2$$

$$\xi = \frac{\delta}{x} dr \quad dr$$

$$a = 1, \quad b = 0, \quad c = \frac{1}{4}$$

$$= \frac{x}{\delta} d\xi$$

$$q = b^2 - 4ac = 1$$

$$r \, dr = \frac{x^2}{\delta^2} \xi \, d\xi$$

$$\int \frac{x}{x^2 + 4} \, dx = \frac{2a + bx}{3qx^3} - \frac{b\xi}{3q} \int \frac{dx}{x^3}$$

$$= -\frac{2}{3\left(1 + \frac{\xi^2}{4}\right)^3}$$

$$J = \frac{16}{3} \pi\rho \gamma^2 v^2 \quad \text{Kinematic momentum} = K = \frac{J}{\rho}$$

$$\left[\frac{3J}{16\pi\rho v^2} \right]^{1/2} = \gamma = \left[\frac{3K}{16\pi v^2} \right]^{1/2}$$

$$u = \frac{2\nu}{x} \left[\frac{3K}{16\pi\nu^2} \right] \left(1 + \frac{\xi^2}{4} \right)^{-2} = \underbrace{\frac{3K}{8\pi\nu x}}_{u_{max}} \underbrace{\left(1 + \frac{\xi^2}{4} \right)^{-2}}_{\text{jet profile differs 2D jet}}$$

$$u_{max} \propto x^{-1}$$

$$v = \underbrace{\frac{\nu}{x} \left[\frac{3K}{16\pi\nu^2} \right]^{1/2}}_{\frac{1}{4x} \left[\frac{3K}{\pi} \right]^{1/2}} \left(\xi - \frac{\xi^3}{4} \right) \left(1 + \frac{\xi^2}{4} \right)^{-2} \quad \xi = \gamma\eta = \left[\frac{3K}{16\pi} \right]^{1/2} \frac{r}{\nu x}$$

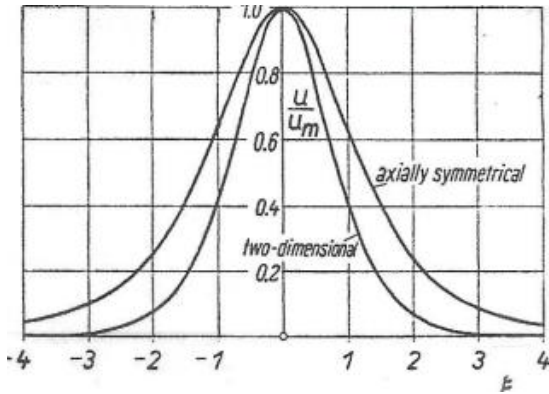


Fig. 9.13. Velocity distribution in a two-dimensional and circular free jet from eqns. (9.44) and (11.15) respectively. For the two-dimensional jet $\xi = 0.275 K^{1/3} y / (\nu x)^{2/3}$, and for the circular jet $\xi = 0.244 K'^{1/2} y / \nu x$. K and K' denote the kinematic momentum J/ρ

$$Q = 2\pi \int_0^\infty ur dr = \frac{V}{s} = 2\pi \int_0^\infty \frac{\nu}{x} \frac{2\gamma^2}{\left(1 + \frac{\xi^2}{4} \right)^2} r dr$$

$$= \underbrace{4\pi\gamma^2 \frac{\nu x^2}{x \gamma^2}}_{4\pi\nu x} \int_0^\infty \frac{\xi}{\left(1 + \frac{\xi^2}{4} \right)^2} d\xi = 4\pi\nu x \left[\frac{-2}{\left(1 + \frac{\xi^2}{4} \right)} \right]_0^\infty$$

$$= -\frac{bx + 2a}{qx} - \frac{b}{q} \int \frac{x dx}{X} = -\frac{2a}{X} = -\frac{2}{\left(1 + \frac{\xi^2}{4} \right)}$$

$$Q = 8\pi\nu x \neq f(K) \text{ and } \dot{m} = 8\pi\mu x$$

$$2D \text{ jet } Q = 3.3019 (Kvx)^{1/3}$$

Interestingly axisymmetric jet $Q = f(x)$ but $\neq f(J)$, i.e., jet with large J/K has large u is narrower vs. jet with small J/K and small u . Small u jet carries more external flow such that both have same Q if both have same v

$$\xi = \gamma\eta = \gamma \frac{r}{x} \Rightarrow r = \frac{x\xi}{\gamma} \quad \text{large } \gamma \text{ small } r \text{ vs. small } \gamma \text{ large } r$$

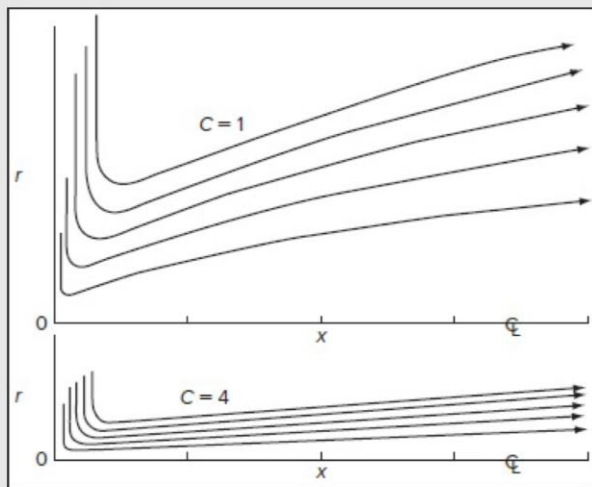


FIGURE 4-42

Streamlines of a round jet for two cases, from the boundary-layer theory of [Eq. \(4-226\)](#).

$C = \gamma$: $\gamma = 1$ not narrow \therefore BL assumptions weak
 $\gamma \geq 4$ narrow and BL assumption OK

$$\frac{J}{\rho} = \int_0^{R_0} u_0^2 (2\pi r) dr = u_0^2 (2\pi) \left[\frac{r^2}{2} \right]_0^{R_0} = u_0^2 \pi R_0^2 = u_0^2 A_0$$

$$A_0 = \pi D^2 / 4$$

$$K = u_0^2 A_0 \Rightarrow u_0 = \left(\frac{K}{A_0} \right)^{1/2}$$

$$Re = \frac{u_0 D}{\nu} = \left(\frac{K}{A_0} \right)^{1/2} \frac{D}{\nu} = \frac{K^{1/2} D}{\left(\frac{\pi D^2}{4} \right)^{1/2} \nu} = \frac{2}{\sqrt{\pi}} \frac{K^{1/2}}{\nu}$$

$$Re_{crit} \approx 30$$

$$\gamma = \left[\frac{3K}{16\pi\nu^2} \right]^{1/2}, \text{ i.e., } \left[\frac{4\gamma^2 16}{3} \right]^{1/2} = \left[\frac{4K}{\pi\nu^2} \right]^{1/2} = \frac{2}{\sqrt{\pi}} \frac{K^{1/2}}{\nu} = Re$$

$$\text{For } \gamma = 4, Re = 18.5$$

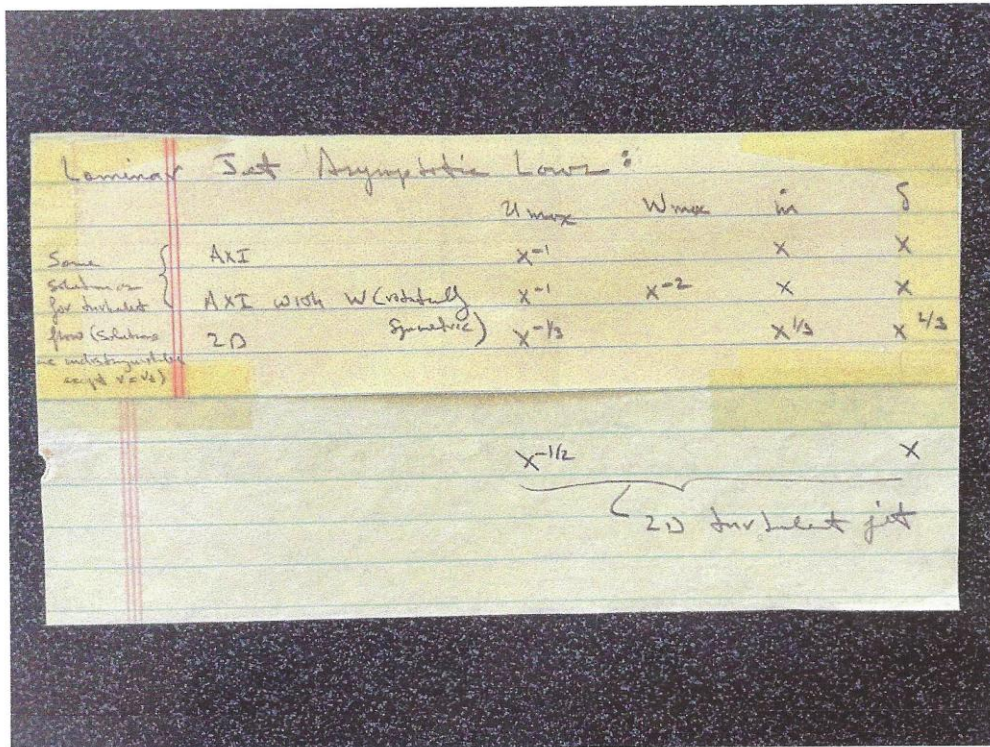
Landau–Squire jet

In [fluid dynamics](#), **Landau–Squire jet** or **Submerged Landau jet** describes a round submerged jet issued from a point source of momentum into an infinite fluid medium of the same kind. This is an exact solution to the incompressible form of the Navier-Stokes equations, which was first discovered by [Lev Landau](#) in 1944^{[1][2]} and later by [Herbert Squire](#) in 1951.^[3] The self-similar equation was in fact first derived by N. A. Slezkin in 1934,^[4] but never applied to the jet. Following Landau's work, V. I. Yatseyev obtained the general solution of the equation in 1950.^[5]

Limiting behaviors

When the force becomes large: the jet in this limiting case is called the [Schlichting jet](#).

Yih, Fluid Mechanics, 1969: Chapter 7 Sections 4.1 (Landau-Squire) and 8 (Schlichting) jets using spherical coordinates.



The swirling round laminar jet

WEI-SHIEN HWANG* and ALLEN T. CHWANG

Department of Mechanical Engineering, Institute of Hydraulic Research, The University of Iowa, Iowa City, IA 52242, U.S.A.

Received 2 November 1990; accepted in revised form 7 July 1991

Abstract. The swirling round laminar jet in an unbounded viscous fluid is investigated in this paper. The axisymmetric laminar jet with a swirling velocity is simulated by a linear-momentum source and an angular-momentum source, both located at the origin. The first-order and the second-order solutions in the far field have been obtained by solving the complete Navier–Stokes equations. It is found that the first-order solution is the well-known round-laminar-jet solution without the swirling velocity obtained by Landau [2] and Squire [3]. The second-order solution represents a pure rotating flow. The swirling velocity predicted by the present solution is compared with that obtained by Loitsyanskii [15] and Görtler [16], who solved the corresponding boundary-layer equations. It is found that the swirling velocity predicted by the present theory is smaller than that obtained from the boundary-layer equations.