2D Far wake of non-lifting bodies

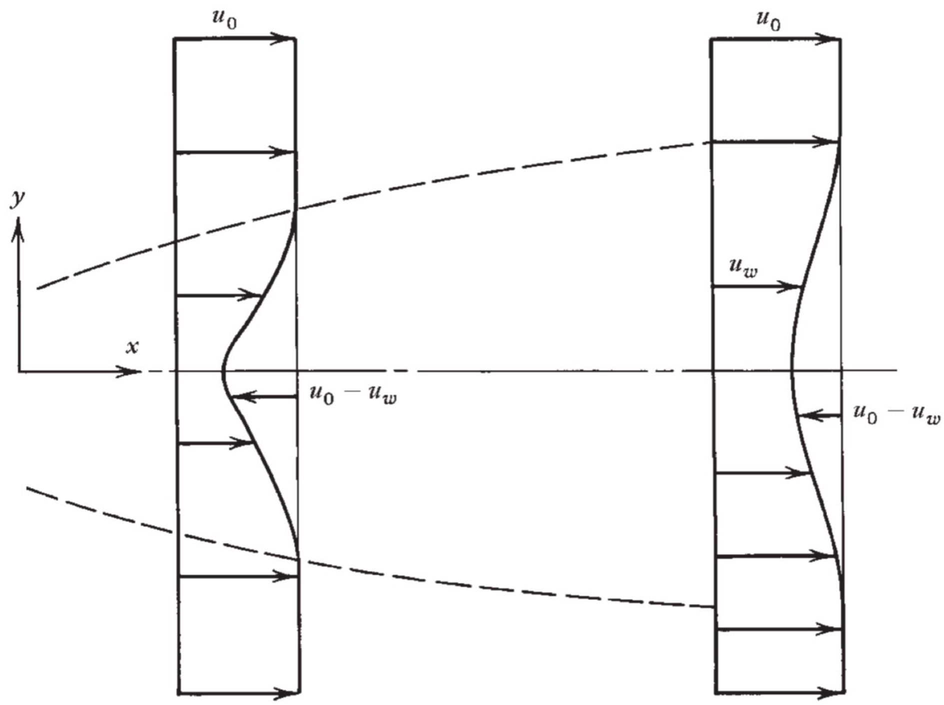

Figure 20.22 Laminar wake of a nonlifting body.

Assume $Re_{body} = \frac{u_0 L}{\nu}$ large, similarly for the velocity defect $u = u_0 - u_w$ and BL equations with $p_x = 0$.

$u_x + v_y = 0 \qquad u\,u_x + v\,u_y = \nu\,u_{yy}$ subscript *w* not shown

$u_w = u_0 - u \quad v_w = v \qquad u = u_0 - u_w$ = velocity defect

$u_{wx} = -u_x$

$u_{wy} = -u_y$

$u_{w_{yy}} = -u_{yy}$

$(u_0 - u)(-u_x) + v(-u_y) = \nu(-u_{yy})$

$(u_0 - u)u_x + vu_y = \nu u_{yy}$

Assume $u = \psi_y, \quad v = -\psi_x$

$$\psi = Ax^p f(\eta) = a(x)f(\eta) \qquad \eta = \frac{y}{Bx^q} = \frac{y}{b(x)}$$

$$a(x) = Ax^p \quad b(x) = Bx^q$$

$$\eta_y = b^{-1} \quad \eta_x = -\frac{\eta b_x}{b}$$

$$a_x = Apx^{p-1} \quad b_x = Bqx^{q-1}$$

$$a_x b = ABpx^{p+q-1} \quad ab_x = ABqx^{p+q-1}$$

A, B, p, q chosen as part of solution.

$$u = \frac{a}{\mathrm{b}} f' \qquad v = -a_x f + af'\left(\eta \frac{ab_x}{b}\right)$$

$$u_x = (a\mathrm{b}^{-1})_x f' + (a\mathrm{b}^{-1})f''\left(-\eta \frac{b_x}{b}\right)$$

$$u_y = (ab^{-2})f''$$

$$u_{yy} = (ab^{-3})f'''$$

$$u_0 u_x = u_0\left[(a_x b^{-1} - ab^{-2}b_x)f' - \left(\eta \frac{ab_x}{b^2}\right)f''\right]$$

$$= u_0\left[\left(\frac{Apx^{p-1}}{Bx^q} - \frac{Ax^p Bqx^{q-1}}{B^2x^{2q}}\right)f' - \left(\frac{\eta Ax^p Bqx^{q-1}}{B^2x^{2q}}\right)f''\right]$$

$$= u_0\left[\left(\frac{Ap}{B}x^{p-q-1} - \frac{Aq}{B}x^{p-q-1}\right)f' - \frac{\eta Aq}{B}x^{p-q-1}f''\right]$$

$$u_0 u_x = u_0 \frac{A}{B} x^{p-q-1} \underbrace{[(p-q)f' - \eta q f'']}_{(1)}$$

$$-uu_x + vu_y = -\left[\left(\frac{Ax^p}{Bx^q}\right)f'\right]\left[\frac{A}{B}x^{p-q-1}\left((p-q)f' - \eta q f''\right)\right]$$

$$+\left(-Apx^{p-1}f + Ax^p f'\left(\eta\frac{Bqx^{q-1}}{Bx^q}\right)\right)\underbrace{\left(\frac{Ax^p}{B^2x^{2q}}\right)}_{\frac{A}{B^2}x^{p-2q}}f''$$

$$= -\frac{A^2}{B^2}x^{2p-2q-1}[(p-q)f'^2 - \eta q f' f'']$$

$$-\frac{A^2}{B^2}px^{2p-2q-1}ff'' + \eta\frac{A^2 q}{B^2}x^{2p-2q-1}f'f''$$

$$= -\frac{A^2}{B^2}x^{2p-2q-1}\underbrace{[(p-q)f'^2 - 2q\eta f'f'' + pff'']}_{(2)}$$

$$u_{yy} = \left(\frac{Ax^p}{B^3x^{3q}}\right)f''' = \frac{A}{B^3}x^{p-3q}f'''$$

$$u_0u_x - uu_x + vu_y = \nu u_{yy}$$

$$u_0B^2x^{2q-1}[(1)] - ABx^{p+q-1}[(2)] = \nu f'''$$

$$F_D = \rho u_0^2\theta = \rho u_0^2\int_{-\infty}^{\infty}\left[\frac{u_w}{u_0} - \left(\frac{u_w}{u_0}\right)^2\right]dy$$

$$= \rho u_0^2\int_{-\infty}^{\infty}u_0^{-2}[u_w u_0 - u_w^2]dy = \rho\int_{-\infty}^{\infty}u_w\,(u_0 - u_w)\,dy$$

$$= \rho\int_{-\infty}^{\infty}(u_0 - u)u\;dy$$

subscript *w* not shown after last equality

$$\eta = yb \qquad d\eta = dy\,\mathrm{b}^{-1} \qquad Bx^q d\eta = dy$$

$u = (ab^{-1})f'$

$(u_0 - (ab^{-1})f')(ab^{-1})f'$

$= u_0(ab^{-1})f' - (ab^{-1})^2 f'^2$

$(u_0 \frac{A}{B} x^{p-q} f' - \frac{A^2}{B^2} x^{2p-2q} f'^2) \times Bx^q = u_0 A x^p f' - \frac{A^2}{B} x^{2p-q} f'^2$

$= Ax^p(u_0 f' - \frac{A}{B} x^{p-q} f'^2)$

$$F_D = \rho A x^p \int_{-\infty}^{\infty} \left[u_0 - \frac{A}{B} x^{p-q} f'\right] f' \, d\eta$$

$p = 0, q = \frac{1}{2}$ independent x *as* $x \to \infty$. Shows as additional restriction is required that $x \to \infty$.

$\frac{u_0 B^2}{2}[-f' - \eta f''] - ABx^{-1/2}[-\frac{1}{2}f'^2 - \frac{1}{2}\eta f f''] = \nu f'''$

Highlight = nonlinear terms drop out Oseen approximation

$$\lim_{x \to \infty} \quad f''' + \frac{u_0 B^2}{2\nu}(f' + \eta f'') = 0$$

$a(x) = Ax^p = A$

$b(x) = Bx^{1/2}$

$d\eta = \frac{dy}{Bx^{1/2}}$

$u = \frac{a}{b} f'$

$f' = \frac{b}{a} u = \frac{Bx^{1/2}}{A} u$

$$F_D = \rho A u_0 \int_{-\infty}^{\infty} f' \, d\eta$$

$$= \rho A u_0 \int_{-\infty}^{\infty} \frac{Bx^{1/2}}{A} \frac{u}{Bx^{1/2}} \, dy = \rho u_0 \int_{-\infty}^{\infty} u \, dy = \rho u_0 Q$$

$Q = \int \underline{V} \cdot \underline{n} dA = \int_{-\infty}^{\infty} u \, dy$ per unit span

$$\text{Let } B = \left(\frac{4\nu}{u_0}\right)^{1/2} \qquad B^2 = \frac{4\nu}{u_0}\cdot\frac{u_0}{2\nu} = 2 \Rightarrow f''' + 2(f' + \eta f'') = 0$$

$$f' = e^{-\eta^2} \qquad f'' = -2\eta e^{-\eta^2} \qquad f''' = -2e^{-\eta^2} + 4\eta^2 e^{-\eta^2}$$

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}$$

$$2(f' + \eta f'') = 2\left(e^{-\eta^2} - 2\eta^2 e^{-\eta^2}\right) \qquad \text{OK}$$

$$F_D = \rho A u_0 \int_{-\infty}^{\infty} e^{-\eta^2} d\eta = \rho A u_0 \pi^{1/2} \Rightarrow A = \frac{F_D}{\rho u_0 \sqrt{\pi}}$$

$$\eta = \frac{y}{\left(\frac{4\nu}{u_0}\right)^{1/2} x^{1/2}} = y\sqrt{\frac{u_0}{4\nu x}}$$

$$\psi = \frac{F_D}{\rho u_0 \pi^{1/2}} \underbrace{\int e^{-\eta^2} d\,\eta}_{\frac{\sqrt{\pi}}{2}\text{erf}(\eta)} = \frac{F_D}{2\rho u_0}\,\text{erf}(\eta)$$

$$\frac{df}{d\eta} = e^{-\eta^2} \qquad f = \int_0^{\eta} e^{-\eta^2} d\,\eta \qquad \text{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2} d\,\eta$$

$$u = u_0 - u_w = \frac{A}{B} x^{1/2} e^{-\eta^2} = \frac{F_D}{\rho u_0 \sqrt{\pi}\left(\frac{4\nu x}{u_0}\right)^{1/2}} e^{-\eta^2}$$

$$u = \left[\frac{F_D^2}{\rho^2 u_0^2 \pi}\cdot\frac{u_0}{4\nu x}\right]^{1/2} e^{-\eta^2}$$

$$u = \left[\frac{F_D^2}{4\pi\rho^2 u_0 \nu x}\right]^{1/2} e^{-\eta^2}$$ velocity defect decays $x^{-1/2}$

$$y = Bx^{1/2} = \left(\frac{4\nu}{u_0}\right)^{1/2} x^{1/2}\eta$$ grows $x^{1/2}$

Momentum thickness wake = constant = $\theta = \frac{F_D}{\rho u_0^2}$

Whereas in the far wake $\underbrace{Q}_{\text{velocity defect } \forall/s} = \frac{F_D}{\rho u_0} = u_0\theta$

It can be shown that $Q = u_0\delta^*$ thus $\delta^* = \theta$ in the far wake

Note we assumed large Re but additionally required $x \rightarrow \infty$, i.e., far wake. It is interesting to note that the same result is obtained using the Oseen equation, which are derived for low Re, i.e., linearized convection.

$$\int_{\delta^*}^{\infty} u_0 \; dy = \int_0^{\infty} u_w \; dy = \int_0^{\infty} u_0 \; dy - \int_0^{\delta^*} u_0 \; dy$$

i.e., $u_0\delta^* = \int_0^{\infty}(u_0 - u_w)\; dy = \int_0^{\infty} u \; dy = Q$ velocity defect flow rate

i.e., in the far wake the velocity defect momentum and displacement thickness are equal