Flow in the wake of flat plate at zero incidence

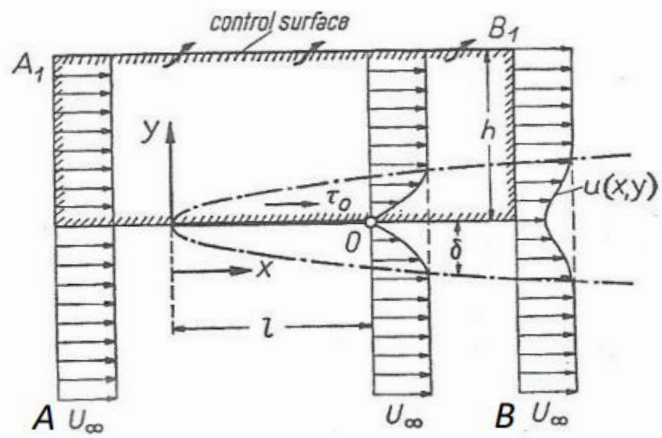


Fig. 9.9. Application of the momentum equation in the calculation of the drag on a flat plate at zero incidence from the velocity profile in the wake

Drag

(1) ψ CV Boundary Layer

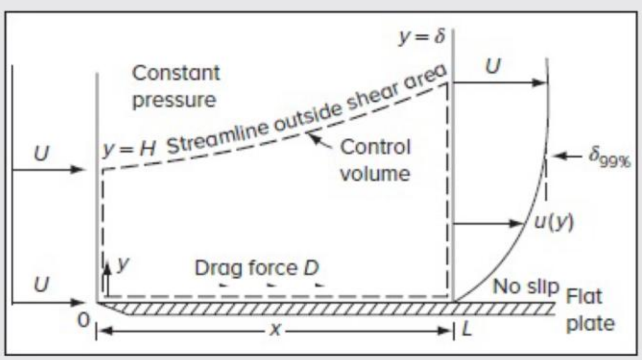


FIGURE 4-1

Definition of control volume for the analysis of flow past a flat plate.

$$\int_{CS} \rho \underline{V} \cdot \underline{n} dA = 0 = -\int_0^H \rho U dy + \int_0^{Y=H+\delta^*} \rho U dy$$

$$UH = \int_0^Y u dy = UY + \int_0^Y (u - U) dy$$

$\delta^* = \int_0^Y (1 - u/U) dy =$ displacement thickness

$$\sum F_x = -D = \int_{CS} \rho u \underline{V} \cdot \underline{n} dA$$

$$= -\int_0^H \rho U(U dy) + \int_0^Y \rho u(u dy)$$

$$D = \rho U^2 H - \int_0^Y \rho u^2 dy$$

$=$ fluid force on plate $= -$ plate force on CV

$D = \rho U^2 \int_0 \frac{u}{U} dy - \int_0 \rho u^2 dy = \int_0^x \tau_w dx$ using continuity

$D/\rho U^2 = \theta = \int_0^Y \frac{u}{U}\left(1 - \frac{u}{U}\right) dy =$ momentum thickness

$C_D = \frac{D}{\frac{1}{2}\rho U^2 x} = \frac{2\theta}{x} = \frac{1}{x}\int_0^x c_f dx$ per unit span

$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$: $c_f = \frac{d}{dx}(x C_D) = 2\frac{d\theta}{dx}$, i.e., $\frac{d\theta}{dx} = c_f/2$

(2) Box CV wake

$$\int_{CS} \rho \underline{V} \cdot \underline{n} = -\int_{A}^{A_1} \rho U dy + \int_{B}^{B_1} \rho u dy + \underbrace{\int_{A_1}^{B_1} \rho v dx}_{\dot{m}_t}$$

$$+ \underbrace{\int_{A}^{B} \rho(-v dx)}_{\dot{m}_b}$$

$$\dot{m}_t + \dot{m}_b = \rho U A A_1 - \int_{B}^{B_1} \rho u dy$$

$$\Sigma F_x = -D = \int_{cs} \rho u \underline{V} \cdot \underline{n} dA = -\int_{A}^{A_1} \rho U(U dy) + \int_{B}^{B_1} \rho u(u dy)$$

$$+ \underbrace{\int_{A_1}^{B_1} \rho u(v dx) + \int_{A}^{B} \rho u(-v dx)}_{\text{assume } u=U\ such\ that = U(\dot{m}_t + \dot{m}_b)}$$

$$D = \rho U^2 A A_1 - \int_{B}^{B_1} \rho u^2 dy - U\left(\rho U A A_1 - \int_{B}^{B_1} \rho u dy\right)$$

$$= U \int_{B}^{B_1} \rho u dy - \int_{B}^{B_1} \rho u^2 dy = \rho \int_{-\infty}^{\infty} u(U - u) dy$$

$$D = \rho U^2 \int_{B}^{B1} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

Same ψ CV with $B = 0$ and $B_1 = Y$. Note for $y > Y, u/U = 1$. Requires integral taken at large distance from the body such that the static pressure at the measurement cross section is the same as that in the undisturbed uniform inflow. Referred to as simplified Betz method, which considers differences in the static pressure between the inflow and the measurement section and is important when the measurement section is too close to body.

$u_x + v u_y = \nu u_{yy}$

$u_1 = U - \mathrm{u(x,y)}$ $\quad u_1 \ll U$ velocity defect

$u = U - u_1 \quad v = v_1 \ll U$

$u_x = -u_{1x} \quad u_y = -u_{1y} \quad u_{yy} = -u_{1yy}$

$(U - u_1)(-u_{1x}) + v_1(-u_{1y}) = v(-u_{1yy})$

$U u_{1x} = \nu u_{1yy}$ to first order

$u_{1y}(0) = 0$

$u_1(\infty) = 0$

Assume similarity for u_1 as per Blasius

$\eta = y\sqrt{U/\nu x} = yU^{1/2}\nu^{-1/2}x^{-1/2}$

$u_1 = cU\sqrt{\frac{L}{x}}g(\eta) \qquad x^{-1/2}$ so that D independent of x

$$2D = b\rho \int_{-\infty}^{\infty} u(U-u)dy = b\rho \int_{-\infty}^{\infty} (U-u_1)(u_1)dy$$

$$= b\rho U \int_{-\infty}^{\infty} u_1 dy$$

$$d\eta = dy\sqrt{U/\nu x} \qquad \frac{d\eta}{dx} = yU^{1/2}\nu^{-1/2}\left(-\frac{x^{-3/2}}{2}\right)$$

$$= b\rho U \int_{-\infty}^{\infty} \left[cU\sqrt{\frac{L}{x}}\ g(\eta)\right]\sqrt{\frac{\nu x}{U}}d\eta$$

$$= b\rho U^2 c\sqrt{\frac{\nu L}{U}}\int_{-\infty}^{\infty} g(\eta)d\eta$$

$$u_{1x} = cU\sqrt{L}\left(-\frac{1}{2}x^{-\frac{3}{2}}\right)g(\eta) + cU\sqrt{\frac{L}{x}}g'[gU^{\frac{1}{2}}v^{-\frac{1}{2}}\left(-\frac{x^{-\frac{3}{2}}}{2}\right)]$$

$$u_{1y} = cUL^{1/2}x^{-1/2}g'(\eta)\sqrt{U/vx}$$

$$u_{1yy} = cUL^{1/2}x^{-1/2}g''Uv^{-1}x^{-1} = \frac{cU^2L^{1/2}x^{-3/2}g''}{v}$$

$$\cancel{c U^2 L^{1/2}} \left(-\frac{1}{2} \cancel{x^{-3/2}}\right) [g + \underbrace{x^{-1/2} y U^{1/2} v^{-1/2}}_{y\sqrt{\frac{U}{vx}}} g']$$

$$= \cancel{c U^2 L^{1/2} x^{-3/2}} g''$$

$$g'' + g/2 + \frac{\eta}{2} g' = 0 \quad g' = 0 \quad \eta = 0 \quad g = 0 \quad \eta = \infty$$

$$g' + \frac{1}{2} \eta g + A = 0$$

$$g'(0) = 0 \Rightarrow A = 0$$

$$g = \exp(-\eta^2/4)$$

where constant of integration (in the form of a coefficient) is taken as unity since u_1 has free coefficient c yet to be determined.

Determine c in terms of the plate drag

$$2D = b\rho U^2 c \sqrt{\frac{\nu L}{U}} \underbrace{\int_{-\infty}^{\infty} g(\eta) d\eta}_{2\sqrt{\pi}}$$

$$= \text{ Blasius solution } = 1.328\ b\rho U^2 \sqrt{\frac{\nu L}{U}}$$

$$\therefore c = .664/\sqrt{\pi}$$

$$\frac{u_1}{U} = \frac{0.664}{\sqrt{\pi}} \left(\frac{L}{x}\right)^{\frac{1}{2}} exp\left(-\frac{1}{4} \frac{y^2 U}{xv}\right) \qquad \propto x^{-1/2}$$

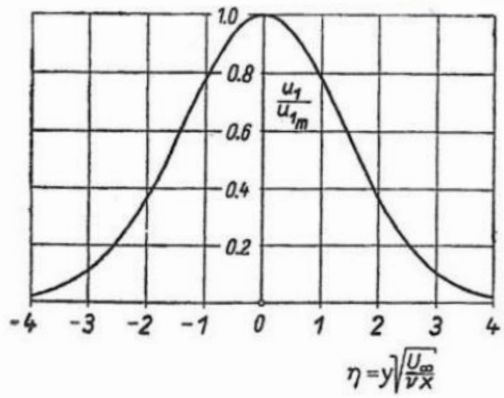
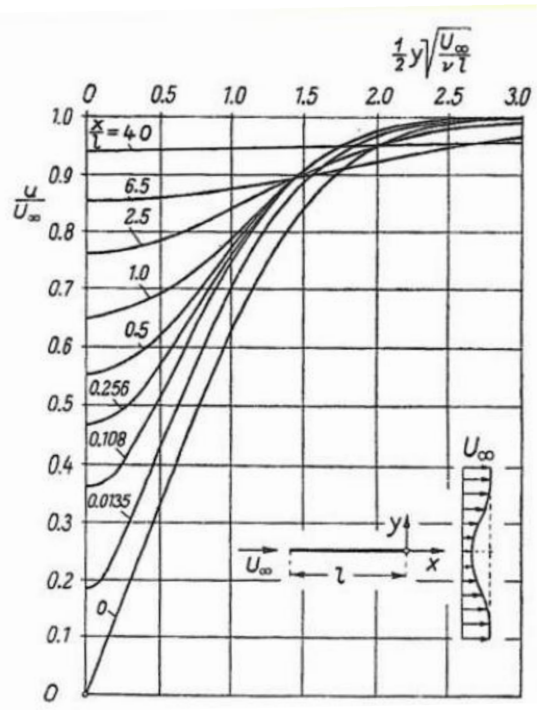


Fig. 9.10. Asymptotic velocity distribution in the laminar wake behind a flat plate, from eqn. (9.35)

Fig. 9.11. Velocity distribution in the laminar wake behind a flat plate at zero incidence ➤



Note that solution only valid in far wake as per assumption $u_1 \ll U$, neglecting higher order terms, and Blasius type similarity. Therefore, $x > 3L$.

Remarkable velocity distribution Gaussian

$Re_{crit} \sim 4$

Boundary layer wake solutions generally require a "far wake" assumption (where the wake is fully developed) because they rely on self-similarity and small velocity deficits to make the governing equations tractable. Jets do not require this, as they are dominated by high-energy momentum at their source and do not depend on weak perturbations of a free stream to maintain their structure.