Transition, Pressure Gradient, and BL Separation

Laminar BL theory altered when transition occurs to turbulent flow or BL separates both of which are affected by $p_x$, especially BL separation. However, turbulent BL may resist separation due to adverse $p_x$.

Transition is f(Re) complicated process not fully understood or predictable based on current understanding of physics or modeling. $Re > Re_{crit}$ transition occurs depending on many factors: free stream turbulence, roughness, $p_x$, vibrations, etc.

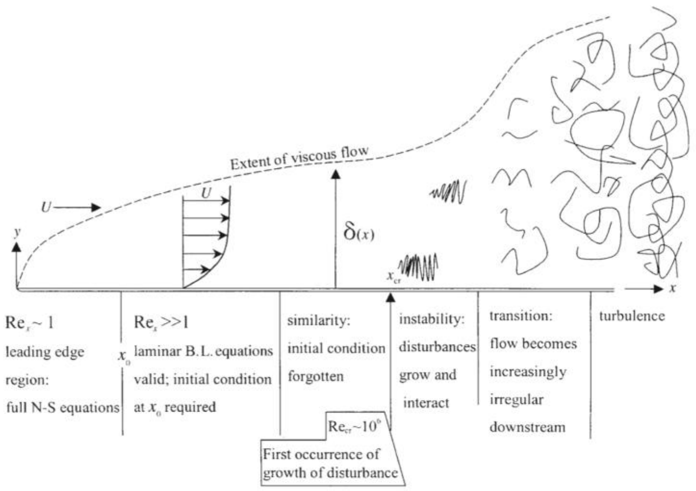



FIGURE 9.10 Schematic depiction of flow over a semi-infinite flat plate. Here, increasing $x$ is synonymous with increasing Reynolds number.

$Re_{crit}$ ↓ when $< u^2 >$ free stream or roughness ↑

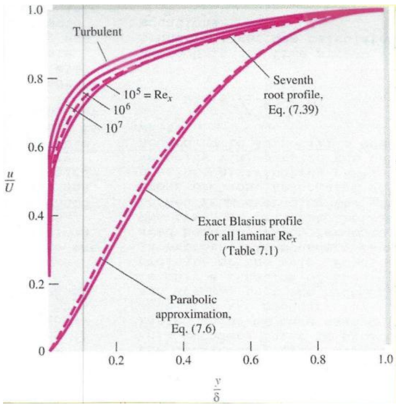
$\tau_w\ turb > \tau_w$ laminar

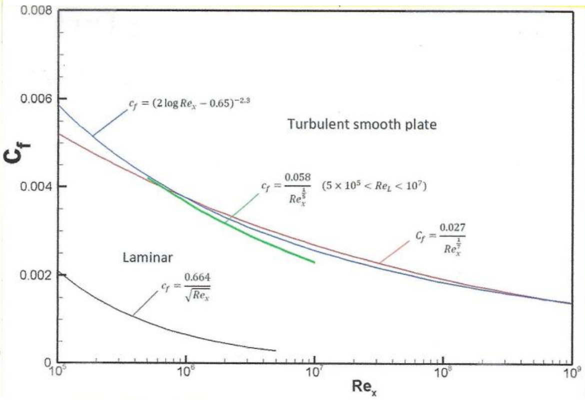
$< uv >_y$ increased diffusion vs. $\nu u_{yy}$ alone

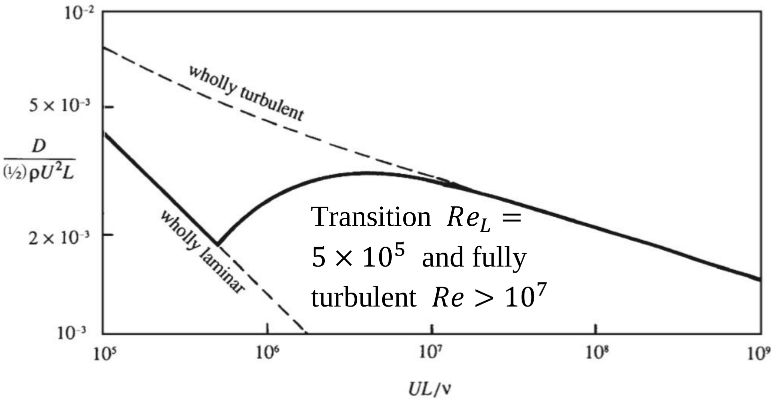
$p_x = 0$

Laminar BL: $\delta \propto x^{1/2}$, $\tau_w \propto x^{-1/2}$, $(U^{3/2})$

Turbulent BL: $\underbrace{\delta \propto x^{6/7}}_{nearly\ linear}$, $\underbrace{\tau_w \propto x^{-1/7}}_{decreases\ more\ slowly}$, $\underbrace{(U^{13/9})}_{nearly\ quadratic}$




The role of $p_x$ in inducing separation is revealed via the BL equations at $y = 0$

$$\mu u_{yy} = p_x$$

1. Accelerating BL $p_x < 0 \Rightarrow u_{yy}|_{wall} < 0$

Since $u_y$ max at wall and decrease to 0 at $\delta$ to merge with $U$

$\therefore u_{yy} < 0$ across whole $\delta$ $\therefore$ no PI

2. Decelerating BL $p_x > 0 \Rightarrow u_{yy}|_{wall} > 0$

Since $u_{yy} < 0$ at $\delta$ $u(y)$ must have PI within $\delta$, which has important implications stability and transition

Note: Blasius $u_{yy} = 0$ at wall.

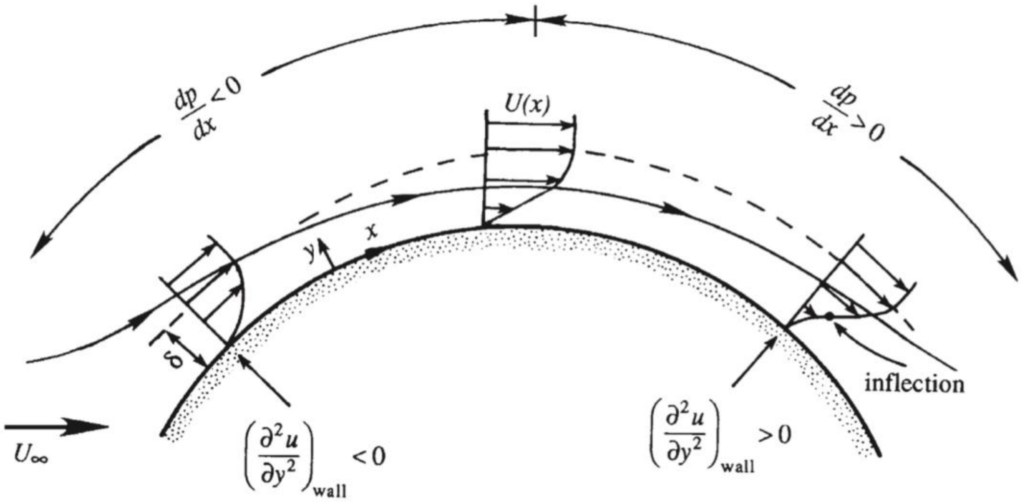



**FIGURE 9.12** Velocity profiles across boundary layers with favorable ($dp/dx < 0$) and adverse ($dp/dx > 0$) pressure gradients, as indicated above the flow. The surface shear stress and stream-wise fluid velocity near the surface are highest and lowest in the favorable and adverse pressure gradients, respectively, with the $dp/dx = 0$ case falling between these limits.

Decelerating outer flow $U_x < 0$ and $p_x > 0$ tends to increase $\delta$, as per FS $v$:

$U_e(x) = ax^n \quad Re_x = \frac{ax^{n+1}}{\nu} \quad -\frac{p_x}{\rho} = U_e U_{e_x} = na^2 x^{2n-1}$

$\psi(x, y) = [\nu x U_e(x)]^{1/2} f(\eta) \quad u = U_e f'(\eta)$

$$\eta = \frac{y}{\delta(x)} = \frac{y}{x}\sqrt{Re_x} = \sqrt{\frac{a}{\nu}}\, x^{\frac{n-1}{2}}$$

$\delta(x) = \left[\frac{\nu x}{U_e}\right]^{1/2} = \left[\frac{\nu x^{1-n}}{a}\right]^{1/2}$ $\delta \uparrow n < 1$ $\delta \downarrow n > 1$ as x increases

$n = 1 \quad \delta =$ constant

$$f''' + \frac{n+1}{2} f f'' - n f'^2 + n = 0$$

$$v = -\psi_x = -\frac{\partial}{\partial x}[[\nu a x^{n+1}]^{1/2} \mathrm{f}(\sqrt{\frac{ax^{n-1}}{\nu}})] =$$

$$-\left[\frac{n+1}{2}(\nu a x^{n-1})^{1/2} f + (\nu a x^{n-1})^{1/2}\frac{n-1}{2}\eta f'\right]$$

Divide by $U_e(x) = ax^n$

$$\frac{v}{ax^n} = \frac{v}{U_e(x)} = -\frac{1}{Re^{1/2}}\left[\frac{n+1}{2}f + \frac{n-1}{2}\eta f'\right]$$

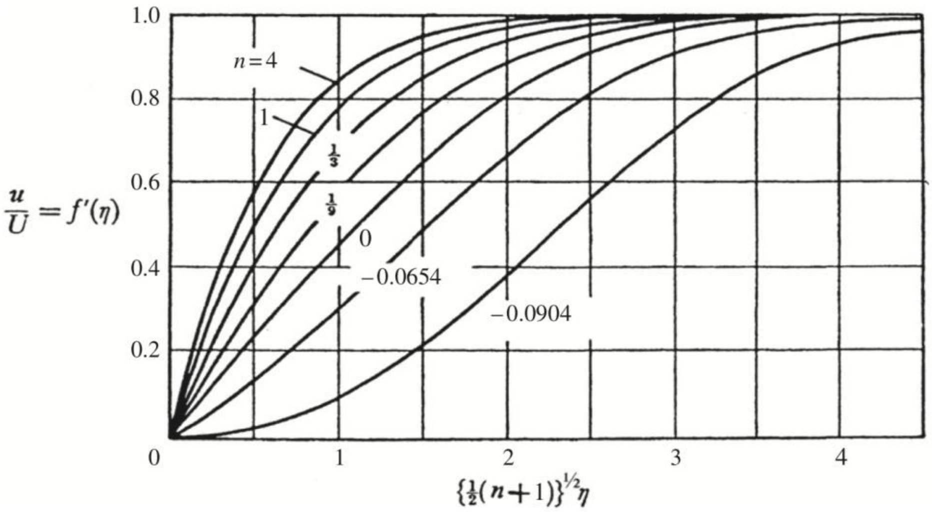



**FIGURE 9.7** Falkner-Skan profiles of stream-wise velocity in a laminar boundary layer when the external stream is $U_e = ax^n$. The horizontal axis is the scaled surface-normal coordinate. The various curves are labeled by their associated value of $n$. When $n > 0$, the free-stream speed increases with increasing $x$, and $\partial^2 u/\partial y^2$ is negative throughout the boundary layer. When $n = 0$ (the Blasius boundary layer), the free-stream speed is constant, and $\partial^2 u/\partial y^2 = 0$ at the wall and is negative throughout the boundary layer. When $n < 0$, the free-stream speed decreases with increasing $x$, and $\partial^2 u/\partial y^2$ is positive near the wall but negative higher up in the boundary layer so there is an inflection point in the stream-wise velocity profile at a finite distance from the surface. *Reprinted with the permission of Cambridge University Press, from: G. K. Batchelor,* An Introduction to Fluid Dynamics, *1st ed. (1967).*

Velocity profiles show that $\eta, f, f' > 0$ for all n in range of interest.

Thus n > 0 accelerating $U_e$ and $v < 0$ such that outer flow driven towards wall thinning BL and prevents separation.

However, when n < 0 decelerating $U_e$ and $v > 0$, e.g., $f < \eta f'$ for n = 0 (Blasius); thus, outer flow displaced from wall, which may lead to separation.

Also, $v(y) = -\int_0^y u_x \, dy$

$p_x = -\rho U U_x \quad U_x < 0 \quad p_x > 0 \Rightarrow -u_x$ in BL also larger $\therefore$ also, larger deceleration $u(y)$ for $y < \delta \Rightarrow v > 0$ larger $\delta \uparrow$ not only $\nu u_{yy}$ but also advection away from surface.

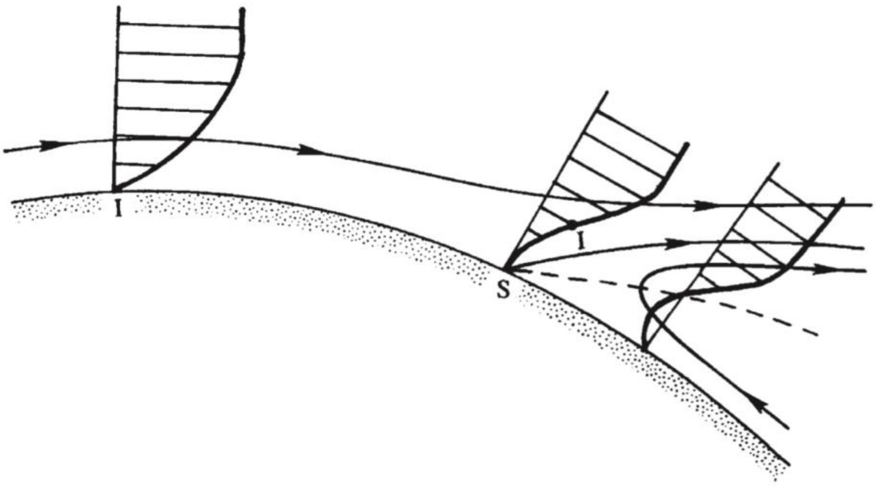



FIGURE 9.13 Streamlines and velocity profiles near a separation point S where a streamline emerges from the surface. The usual boundary-layer equations are not valid downstream of S. The inflection point in the stream-wise velocity profile is indicated by I. The dashed line is the locus of $u = 0$.

For $p_x > 0$ BL flow decelerates, $\delta \uparrow$ and PI within BL. When $p_x$ strong enough and acts over sufficient distance, the flow separates, and a region of reverse flow develops near the wall. The point S at which forward flow meets reverse flow is a local stagnation point = the separation point. Fluid elements approach S from either side; thus, a separation $\psi$ emerges from the surface at S. Furthermore, $\tau_w$ changes sign at S since surface flow changes direction.

$\therefore u_y|_{\text{wall}} = 0$ at S

Note BL assumptions and equations no longer valid; therefore, viscous/inviscid interaction and streamwise diffusion important

## EXAMPLE 10.8

Using a third-order two-dimensional power-series expansion near a flat-plate boundary layer's separation point, $x = x_s$ and $y = 0$, determine how the stream function $\psi(x,y)$ depends on $\partial p/\partial x$ and $\beta_s$, the angle the separating streamline makes with the horizontal surface as shown in Figure 10.17.

### Solution

A third-order power series expansion for $\psi(x,y)$ is:

$$\psi(x,y) = a_0 + a_1x' + a_2y + a_3x'^2 + a_4x'y + Ay^2 + a_5x'^3 + a_6x'^2y + Bx'y^2 + Cy^3.$$

where $x' = x - x_s$, and $a_0$ through $a_6$, $A$, $B$, and $C$ are undetermined constants. This stream function must satisfy the no-slip boundary condition, $u = v = 0$ on $y = 0$, so $\partial\psi/\partial y = -\partial\psi/\partial x = 0$ on $y = 0$. These two conditions cause $a_1$ through $a_6$ to be zero, and if $\psi = 0$ defines the plate surface, then the stream function reduces to $\psi(x,y) = Ay^2 + B(x - x_s)y^2 + Cy^3$. In addition, the surface shear stress, $\tau_w$, is zero at the separation point, so:

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0,x=x_s} = \mu\left(\frac{\partial^2\psi}{\partial y^2}\right)_{y=0,x=x_s} = (2A + 2B(x - x_s) + 6Cy)_{y=0,x=x_s} = 2A = 0,$$

and this leaves:

$$\psi(x,y) = B(x - x_s)y^2 + Cy^3.$$

In the vicinity of the separation point, this stream function $\psi(x,y)$ must satisfy two additional conditions. The first comes from the limiting form of (9.1) as $y \to 0$ (see Example 9.1), which for the present coordinate system and stream function is:

$$(\partial p/\partial x)_{y=0} = \mu(\partial^2 u/\partial y^2)_{y=0} = \mu(\partial^3\psi/\partial y^3)_{y=0},$$

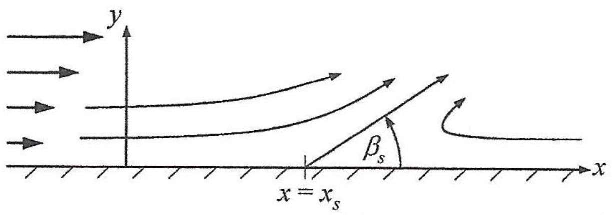



**FIGURE 10.17** Streamline pattern near the separation point ($x = x_s$, $y = 0$) on a flat surface.

and this implies $C = (1/6\mu)(\partial p/\partial x)$. The second condition is that the zero-streamline must leave the surface at an angle $\beta_s$ with respect to the downstream direction. The zero-streamline is given by $\psi(x,y) = 0$, which implies:

$$0 = B(x - x_s)y^2 + \frac{1}{6\mu}\left(\frac{\partial p}{\partial x}\right)y^3, \quad \text{or} \quad -\frac{1}{6\mu}\left(\frac{\partial p}{\partial x}\right)y = B(x - x_s), \quad \text{or} \quad -\frac{1}{6\mu}\left(\frac{\partial p}{\partial x}\right)\frac{dy}{dx} = B.$$

So, with $dy/dx = \tan\beta_s$, the final form for the stream function expansion is:

$$\psi(x,y) = \frac{y^2}{6\mu}\left(\frac{\partial p}{\partial x}\right)(y - (x - x_s)\tan\beta_s).$$

Thus for boundary layer separation from a flat surface, the angle of the separating streamline may be independent of the local pressure gradient. And, when the flow is in the positive $x$-direction upstream of the separation point (i.e. $\psi > 0$ for $y > 0$), this stream function only makes sense when $\partial p/\partial x$ is locally positive, an adverse pressure gradient.

Separation is a complex process that is usually 3D and depending on $Re$ unsteady.

External flow geometry: bluff, slender, sharp edge vs. smooth surface; form/pressure vs. friction $C_D$.

Internal flow geometry: convergent/divergent; bends; and other minor losses

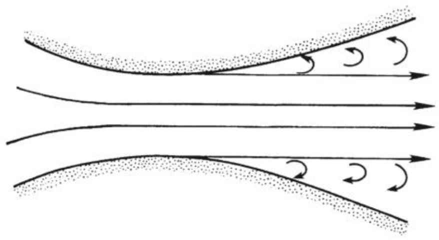


FIGURE 9.15 Separation of flow in a divergent channel. Here, an adverse pressure gradient has led to boundary-layer separation just downstream of the narrowest part of the channel. Such separated flows are instable and are exceedingly likely to be unsteady, even if all the boundary conditions are time independent.

Bluff-body: higher p forebody and low $p \sim$ constant afterbody called base pressure in separation region. At higher $Re$ size of the wake depends laminar vs. turbulent flow. Slender body: LE & TE separations.

Low Re even bluff body wake can be steady, e.g., circular cylinder. $0.4 < Re < 40$ steady vortex in wake. At higher Re, BL on cylinder and smooth surface separation $f(Re)$, e.g., drag crisis $S = 82°\text{vs.} 125°$

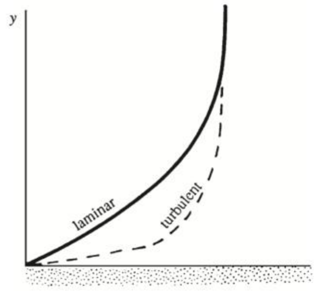



FIGURE 9.14 Nominal comparison of laminar and mean-turbulent, stream-wise velocity profiles in a boundary layer. Here the primary differences are the presence of higher speed fluid closer to the surface and greater surface shear stress in the turbulent layer.

S laminar less sensitive *Re*. Turbulent BL more resistant separation due fuller profile higher momentum flow near wall.