

Axisymmetric Boundary Layers

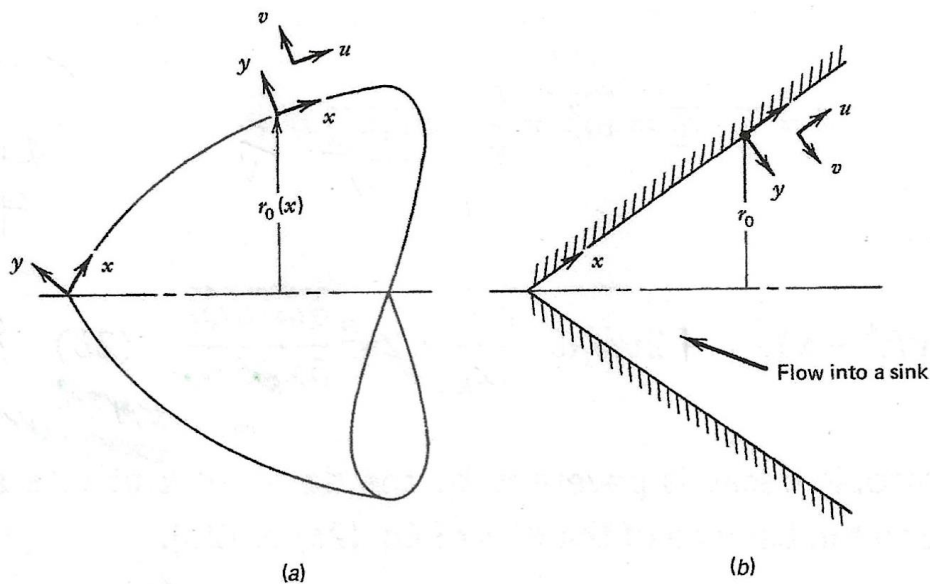


Figure 20.20 Nomenclature for Mangler's transformation: (a) axisymmetric boundary layer and (b) flow into a conical passage.

Assume no swirl and δ is small vs. both longitudinal and transverse radius of curvature. Boundary layer equations derived by Mangler (1945), which differ from 2D planar case by the appearance in the continuity equation of r_0 :

$$\frac{\partial}{\partial x}(r_0 u) + r_0 \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$0 = \frac{\partial p}{\partial y}$$

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = u_e \frac{du_e}{dx}$$

Mangler also gave a mathematical transformation that transforms the axisymmetric problem into an equivalent plane problem.

$$x' = \int_0^x \left(\frac{r_0}{L}\right)^2 dx \quad y' = \frac{r_0 y}{L}$$

$$u' = u \quad U'(x') = U(x) \quad v' = \frac{L}{r_0} \left(v + \frac{yu}{r_0} \frac{dr_0}{dx} \right)$$

Where L is an arbitrary length scale. Substitution into axisymmetric boundary layer equations $\left[\frac{\partial x'}{\partial x} = \left(\frac{r_0}{L}\right)^2\right]$ and $\left[\frac{\partial y'}{\partial x} = \frac{y}{L} \frac{\partial r_0}{\partial x}\right]$ produces the plane flow boundary layer equations:

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0$$

$$u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} = U' \frac{\partial U'}{\partial x'} + \nu \frac{\partial^2 u'}{\partial y'^2}$$

A solution of an axisymmetric boundary layer may be found by considering an equivalent plane boundary layer via Mangler's transformation. When r_0 is constant the transformation is trivial.

As an example, consider axisymmetric stagnation point on a blunt body: Homann flow. The inviscid velocity near the stagnation point is given by,

$$\frac{u_e}{u_0} = \frac{x}{L} \quad u_0 = \alpha u_\infty$$

L = characteristic body dimension and α depends on the shape of the body. For $r_0 = x$,

$$\frac{x'}{L} = \int_0^{x/L} \left(\frac{r_0}{L}\right)^2 d\frac{x}{L} = \frac{1}{3} \left(\frac{x}{L}\right)^3$$

$$\frac{y'}{L} = \frac{r_0 y}{L L} = \frac{x y}{L L}$$

Which define the point x' , y' in the plane flow, which is equivalent to the point x , y in the axisymmetric flow. Next transform the external velocity,

$$u_e' = u_e$$

$$\frac{u_e'}{u_0} = \frac{x}{L} = \left(3 \frac{x'}{L}\right)^{1/3}$$

The equivalent plane flow is, $u_e \approx x'^{1/3}$ where the reference velocity constant is unchanged, but the length constant is $\frac{L}{3} = L'$.

The solution to a boundary layer obeying the $\frac{x'}{L}$ and $\frac{y'}{L}$ is the Falkner-Skan flow for $m=1/3$ ($\theta_{1/2} = 45^\circ$ wedge). Assuming that this solution is known, the u velocity at the point x , y in the axisymmetric stagnation point flow would be,

$$\frac{u(x, y)}{u_0} = f' \left[\frac{y'}{L} \rightarrow \frac{x}{L} \frac{y}{L'}, \frac{x'}{L} \rightarrow \left(\frac{1}{3} \frac{x'}{L} \right)^3 \right]$$

A corresponding formula for v is found from $v' = \frac{L}{r_0} \left(v + \frac{yu}{r_0} \frac{dr_0}{dx} \right)$. An axisymmetric stagnation point is 80% as thick as a plane stagnation point.

The flow toward a axisymmetric stagnation point is a special case of the streaming flow over a cone of a given angle. Since the external flow over a cone obeys $u_e \propto x^n$ and the surface position is $r_0 = x \sin \theta_{1/2}$, Mangler's transformation produces $u' \propto x'^{n/3}$, which is one of the Falkner-Skan solutions. Unfortunately, there is no simple relation between the cone angle θ and the exponent n . Whitehead and Canetti (1950) prove a graphical relationship.

Three-dimensional boundary layers: flow driven by both p_x and p_z

$$u_x + v_y + w_z = 0$$

$$uu_x + vu_y + wu_z = -\frac{p_x}{\rho} + vu_{yy}$$

$$uw_x + vw_y + ww_z = -\frac{p_z}{\rho} + vw_{yy}$$

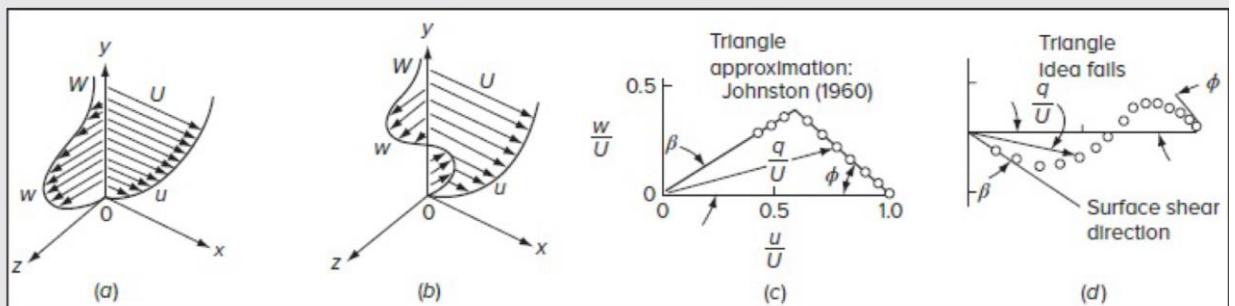


FIGURE 4-45

Unidirectional and bidirectional skewing in three-dimensional pressure-driven boundary layers: (a) unidirectional skewing; (b) bidirectional skewing; (c) unidirectional hodograph; (d) bidirectional hodograph.

Complex crossflow such as (b): death of 3D integral methods motivated 3D differential BL methods:

Three-dimensional turbulent boundary layers

- [J. Nash, V. C. Patel](#)
- Published 1972
- Physics, Engineering

Thick BL and separation led to rapid extensions differential methods to modern CFD: RANS, HRLES, LES, DNS.

Boundary layer with constant transverse pressure gradient: flat plate BL with parabolic free stream.

Flat plate with LE at $x = 0$ and inflow at angle θ_0

For $x > 0$ & $y \rightarrow \infty$

$$u = u_e = \text{constant} \quad p_x = 0$$

$$w = w_e = u_e(a + bx) \quad -\frac{p_z}{\rho} = u_e w_{e,x} = u_e^2 b$$

$b =$ magnitude transverse p_z

$$\text{External flow } \underline{w} = w_y \hat{j} \quad w_y = -\frac{\partial w}{\partial x} = -u_e b$$

For $b = 0$, $\theta = \theta_0 = \tan^{-1} \left(\frac{w_e}{u_e} \right) = \tan^{-1} a$, which shows the effect of sweep on Blasius BL

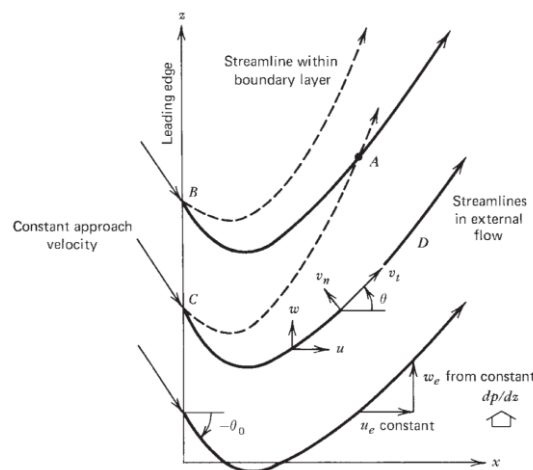


Figure 20.30 Plan view of the boundary layer on a plate with a constant transverse pressure gradient.

$$\frac{dz_e}{dx} = \frac{w_e}{u_e} = a + bx \quad z_e = ax + \frac{1}{2}bx^2 + c$$

$$c = \text{constant} = z_e|_{LE}$$

Concept of region of influence: particles that pass through c influence region CAD . 3D BL method must account for spreading lateral influence.

BL equations:

$$u_x + v_y = 0$$

$$uu_x + vu_y = \nu u_{yy}$$

$$uw_x + vw_y = bu_0^2 + \nu w_{yy}$$

continuity and x-momentum $\neq f(w) =$ Blasius BL problem

$$\frac{u}{u_e} = f'(\eta) \quad \eta = y \sqrt{\frac{u_e}{\nu x}}$$

$f(\eta)$ solution Blasius equation: $ff'' + 2f''' = 0$

Sweep independence principle: u, v solved independent w

$$\text{Assume: } \frac{w}{u_e} = \underbrace{\frac{w_e(x)}{u_e}}_{f'(\eta)} + bx h(\eta)$$

Blasius component scaled to match $w_e(x)$

Substitution in w-momentum equation

$$\left. \begin{aligned} h'' + \frac{1}{2}fh' - f'h + 1 - f'^2 &= 0 \\ h(0) = h(\infty) &= 0 \end{aligned} \right\} \text{Similarly, Blasius equation solved numerically}$$

Solutions use normal and tangential coordinates to outer flow and normalize with outer flow velocity magnitude $V_\infty = [u_e^2 + w_e^2]^{1/2}$

$$\frac{v_t}{V_\infty} = f'(\eta) + \frac{1}{2} \sin 2\theta (\tan \theta - \tan \theta_0) h(\eta)$$

$$\frac{v_n}{V_\infty} = \frac{1}{2} \cos^2 \theta (\tan \theta - \tan \theta_0) h(\eta)$$

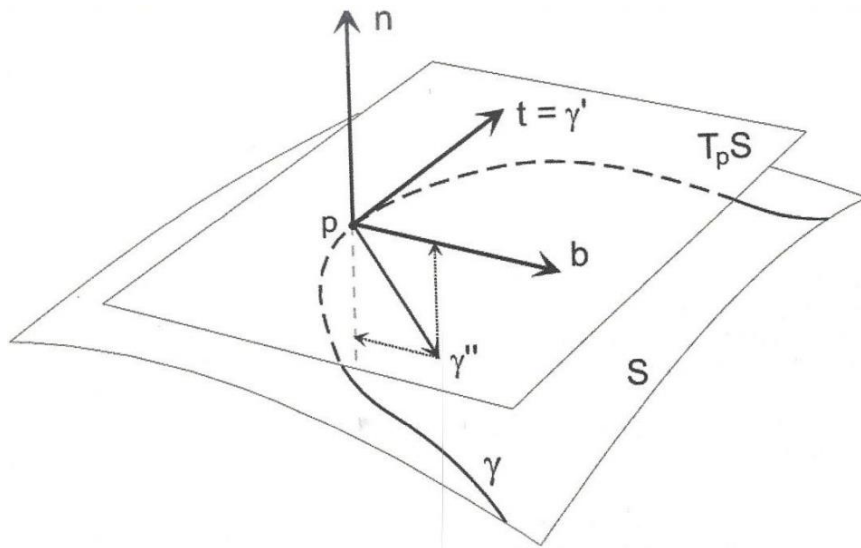
$\theta =$ local inviscid ψ angle $\tan^{-1}(w_e/u_e)$

(1) flat plate with sweep angle $\theta = \theta_0$ and $b = 0$

Blasius profile and no secondary flow i.e. $v_n = 0$

If ψ_e coincides with the surface geodesic curves $\psi_{BL} = \psi_e$ and no secondary flow

Geodesic curve: curve whose tangent vectors remain parallel if they are transported along it



(2) $\theta \neq \theta_0$ and $b \neq 0$

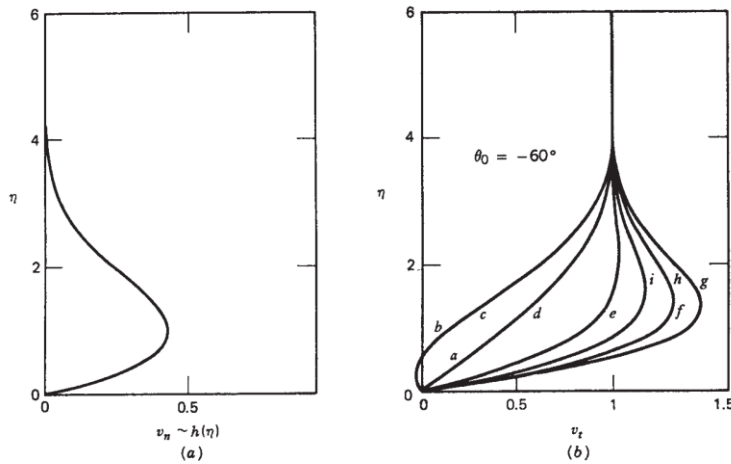


Figure 20.31 Typical velocity profiles: (a) transverse profiles $h(\eta)$; and (b) tangential profiles for various flow angles: a, $\theta = -60^\circ$; b, $\theta = -40^\circ$; c, $\theta = -20^\circ$ (same curve as b); d, $\theta = 0^\circ$ (same as a); e, $\theta = 20^\circ$; f, $\theta = 40^\circ$; g, $\theta = 60^\circ$; h, $\theta = 80^\circ$ (same as f); i, $\theta = 90^\circ$.

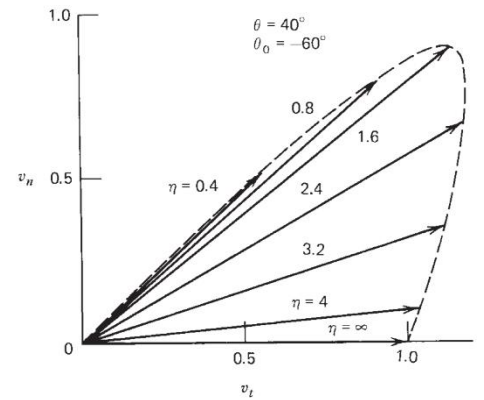


Figure 20.32 Polar diagram of velocity profile with overshoot.

v_n not shown since $\propto h(\eta)$

θ			
-60	a	Blasius	
-40	b		$v_t < 0$ near $\eta = 0$, but not reverse flow
-20	c	\Rightarrow	v_n large such that flow > 0 in x direction
0 = a	d	Blasius	
20	e		v_t exhibits overshoot in upper BL occurs frequently in 3D BL in these regions (similar as Stokes 2 nd problem for oscillating outer flow) net transverse viscous force some direction pressure force. viscous force dies out at large η
40	f		
60	g	\Rightarrow	
80 = f	h		
90	I		

Note: flow never actually separates

$$z_s = ax + \frac{1}{2}bx^2 \left[1 + \frac{h'(0)}{f''(0)} \right] + C_0 \text{ also parabolic and differs } z_e \text{ by}$$

$$1 + \frac{h'(0)}{f''(0)} \approx 1 + \frac{1}{3}$$