2D mixing layer: free shear layer

Two parallel streams, U_1 (upper) and U_2 (lower) meet at $x = 0$. Progression downstream smooths discontinuity due to jump into S shaped velocity profile is free-shear layer. Assume (1) p constant (upper/lower layer), i.e., $p_x = 0$ and u_1 and u_2 uniform streams such that $y = 0$ dividing ψ and (2) BL assumptions. Blasius type similarity variables $\eta_\alpha = \left(\frac{U_1}{\upsilon_\alpha x}\right)^{1/2} y$ and $\psi = (\nu_\alpha U_1 x)^{1/2} f_\alpha(\eta_\alpha)$ result in Blasius-type equation for each layer but with different BC.

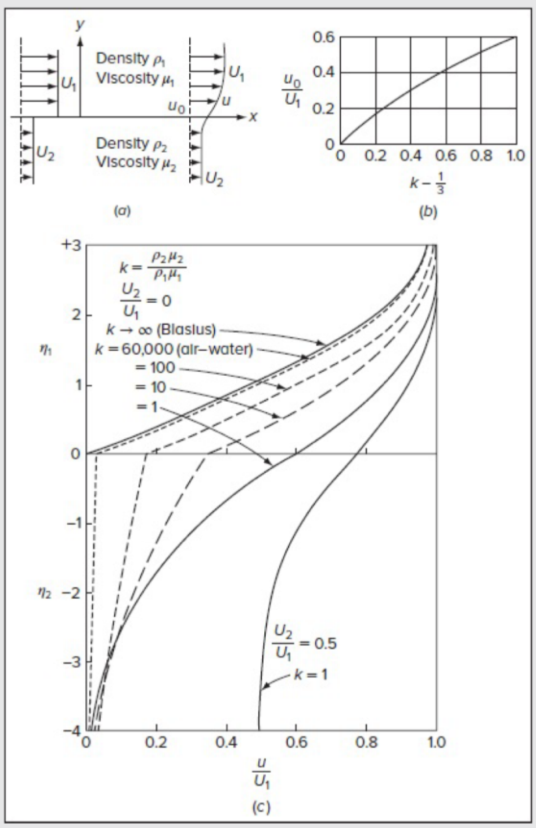


FIGURE 4-22
Velocity distribution between two parallel streams with dissimilar properties: (*a*) geometry; (*b*) velocities at the interface ($U^2 = 0$); (*c*) representative velocity profiles. [*After Lock (1951), by permission of The Clarendon Press, Oxford*.]

Note: at $y = 0$ $\mu_1 u_{1y} = \mu_2 u_{2y}$ such that μ itself important physical property not just $\nu = \mu/\rho$.

$u_1 > 0$ whereas $u_2 \geq 0$

Blasius Similarity

$$\psi_\alpha = (\nu_\alpha U_1 x)^{1/2} f_\alpha(\eta_\alpha) = g_\alpha(x) f_\alpha$$

$$\eta_\alpha = \left(\frac{U_1}{\nu_\alpha x}\right)^{1/2} y = h_\alpha(x)\, y$$

$$\eta_{\alpha_x} = y\, h_{\alpha_x} = \frac{\eta_\alpha}{h_\alpha} h_{\alpha_x}$$

$$\eta_{\alpha_y} = h_\alpha \quad \eta_{\alpha_{yy}} = 0$$

$$u u_x + v u_y = \nu u_{yy}$$

$$u_\alpha = U_1 f'_\alpha = \psi_{\alpha y}$$

$$\psi_{\alpha y} = [\nu_\alpha U_1 x]^{1/2} f'_\alpha \eta_{\alpha y} = [\nu_\alpha U_1 x]^{1/2} \left(\frac{U_1}{\nu_\alpha x}\right)^{1/2} f'_\alpha = U_1 f'$$

$$v_\alpha = -\psi_{\alpha_x} = -\left(g_{\alpha_x} f_\alpha + g_\alpha f_{\alpha_x}\right) = -\left(g_{\alpha_x} f_\alpha + g_\alpha f'_\alpha \eta_{\alpha_x}\right)$$

$$g_\alpha(x) = [\nu_\alpha U_1]^{1/2} x^{1/2} \quad g_{\alpha_x} = \frac{1}{2}[\nu_\alpha U_1]^{1/2} x^{-1/2} = \frac{1}{2}\left[\frac{\nu_\alpha U_1}{x}\right]^{1/2}$$

$$h_\alpha = \left[\frac{U_1}{\nu_\alpha}\right]^{1/2} x^{-1/2} \quad h_{\alpha_x} = -\frac{1}{2}\left[\frac{U_1}{\nu_\alpha}\right]^{1/2} x^{-3/2}$$

$$\frac{h_{\alpha_x}}{h_\alpha} = \frac{-\frac{1}{2}\left[\frac{U_1}{\nu_\alpha}\right]^{1/2} x^{-3/2}}{\left[\frac{U_1}{\nu_\alpha}\right]^{1/2} x^{-1/2}} = -\frac{1}{2} x^{-1}$$

$$-v_\alpha = \frac{1}{2}\left[\frac{\nu_\alpha U_1}{x}\right]^{1/2} f_\alpha + [\nu_\alpha U_1 x]^{1/2} f'_\alpha \eta_\alpha \left(-\frac{1}{2x}\right)$$

$$v_\alpha = \frac{1}{2}\left[\frac{\nu_\alpha U_1}{x}\right]^{1/2} [\eta_\alpha f'_\alpha - f_\alpha]$$

$$u_{\alpha_x} = U_1 f_\alpha'' \eta_{\alpha_x} = u_1 f_\alpha'' \left(-\frac{\eta_\alpha}{2x}\right) \qquad \eta_{\alpha_x} = -\frac{\eta}{2x}$$

$$u_{\alpha_y} = u_1 f_\alpha'' \eta_{\alpha_y} = u_1 f_\alpha'' \left[\frac{U_1}{\nu_\alpha x}\right]^{1/2}$$

$$u_{\alpha_{yy}} = u_1 f_\alpha''' \left(\eta_{\alpha_y}\right)^2 = u_1 f_\alpha''' \left[\frac{U_1}{\nu_\alpha x}\right]$$

Drop α

$$u_1 f' \left[U_1 f'' \left(-\frac{\eta}{2x}\right)\right] + \frac{1}{2}\left[\frac{\nu U_1}{x}\right]^{1/2} [\eta f' - f] u_1 f'' \left[\frac{U_1}{\nu x}\right]^{1/2}$$

$$= \nu u_1 f''' \left[\frac{U_1}{\nu x}\right]$$

$$U_1^2 \left(-\frac{\eta}{2x}\right) f' f'' + \frac{1}{2}\left(\frac{U_1^2}{x}\right) [\eta f' - f] f'' = \frac{U_1}{x} f'''$$

$$-\frac{\eta}{2} f' f'' + \frac{1}{2}(\eta f' f'' - f f'') = f''' \qquad 2f''' + f f'' = 0$$

$2f_\alpha''' + f_\alpha f_\alpha'', = 0$ ODE solved numerically subject BC.

BC: $u_1 = u_2, \quad v_1 = v_2 = 0, \quad \mu_1 u_{1y} = \mu_2 u_{2y} \quad y = 0$

$u_1 = U_1, \quad u_2 = U_2 \quad \text{as } y \to \pm\infty$

$$f_1'(0) = f_2'(0), f_1(0) = f_2(0) = 0, \mu_1 u_1 f_1'' \left[\frac{U_1}{\nu_1 x}\right]^{1/2} = \mu_2 u_1 f_2'' \left[\frac{U_1}{\nu_2 x}\right]^{1/2}$$

i.e., $f_1''[\mu_1\rho_1]^{1/2} = f_2''[\mu_2\rho_2]^{1/2} f_1''(0) = \left[\frac{\rho_2\mu_2}{\rho_1\mu_1}\right]^{1/2} f_2''(0) = A^{1/2} f_2''(0)$

$$f_1'(\infty) = U_1, \quad f_2'(-\infty) = \frac{U_2}{U_1}$$

$A = 5.97 \times 10^4 = \frac{water}{air}$ shown with $U_2 = 0$ (upper layer air and lower layer water). Also, other A values down to 1 for which $\frac{u}{U_1}(0) = 5.8$ at $y = 0$. Discontinuity smoothed by μ as move downstream to S-shaped free shear layer. Assume no mass transfer between fluids, i.e., immiscible. Most practical cases: air/water and $\rho_1\mu_2 = \rho_2\mu_1$, i.e., same fluid. $A \uparrow$ lower fluid moves more slowly; air/water slow wind driven motion.

$A \neq 1, \quad U_2 = 0$: jump velocity gradient at $y = 0$

$A = 1, \quad U_2 = 0$:

(1) asymmetric, $\frac{u}{U_1}(0) > 0.5$ due differences convective deceleration between upper/lower.

(2) $\frac{f_2(-\infty)}{\sqrt{2}} = -0.619$ = flat plate at $-\infty$ with BL blown off, as per Flat Plate with Wall Suction or Blowing.

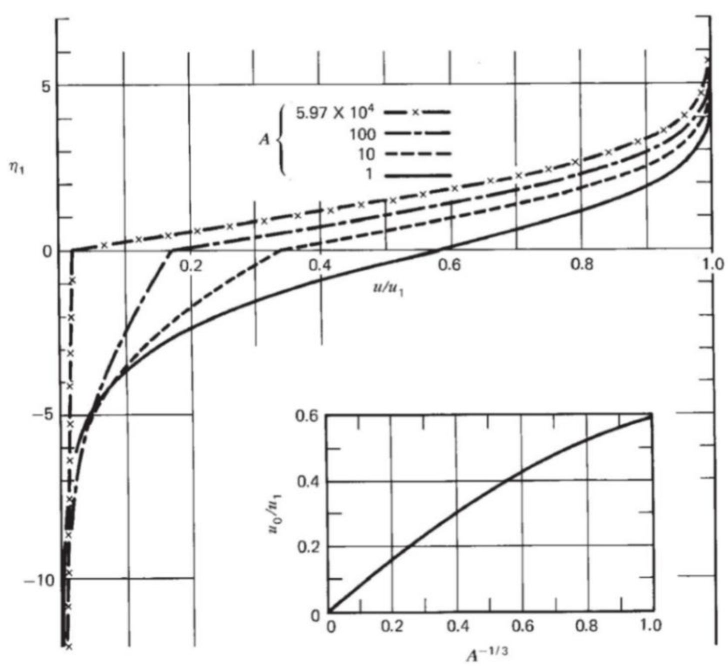


Figure 20.23 Velocity profiles between shear layers. $A = \rho_1\mu_1/\rho_2\mu_2$. Inset shows centerline velocity as a function of A. Reprinted by permission from Lock (1951) Oxford University Press.

Additional discussion

(1) For $\rho_1 \neq \rho_2$, $\mu_1 \neq \mu_2$ jump fluid properties and discontinuity at interface. 20° 1 atm

	ρ [kg/m^3]	μ [kg/ms]	ν [m^2/s]
air	1.205	1.800E-05	1.500E-05
water	1000	1.003E-03	1.005E-06

	ρ	μ	ν	$\rho\mu$	$\sqrt{\rho\mu}$
water/air	832	56	0.067	46592	216

∴ water/air jump condition mainly due ρ which is reason sharp interface methods often; only consider ρ jump and smooth μ

(2) $v_2 = \frac{1}{2}\left[\frac{\nu_2 U_1}{x}\right]^{1/2}[\eta_2 f_2' - f_2] \qquad \eta_2 = \left[\frac{U_1}{\nu_2 x}\right]^{1/2} y$

$$v_2(-\infty) = \frac{1}{2}\left[\frac{\nu_2 U_1}{x}\right]^{1/2}\left[\left(\frac{U_1}{\nu_2 x}\right)^{1/2}\frac{U_2}{U_1} - f_2(-\infty)\right] = -\frac{1}{2}\left[\frac{\nu_2 U_1}{x}\right]^{1/2} f_2(-\infty)$$

$$\frac{v_2(-\infty)}{\frac{1}{2}\left[\frac{\nu_2 U_1}{x}\right]^{1/2}} = f_2(-\infty) = -0.619$$

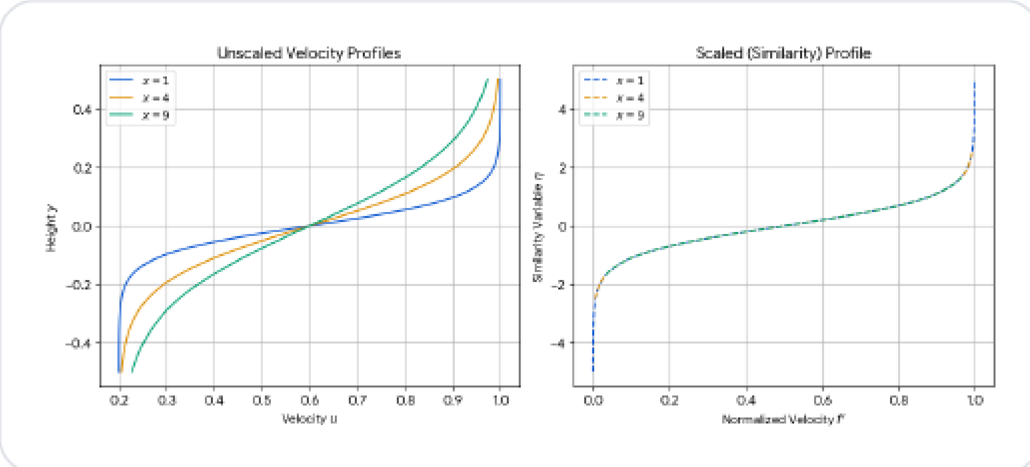
(3) Interesting compare diffusion vortex sheet

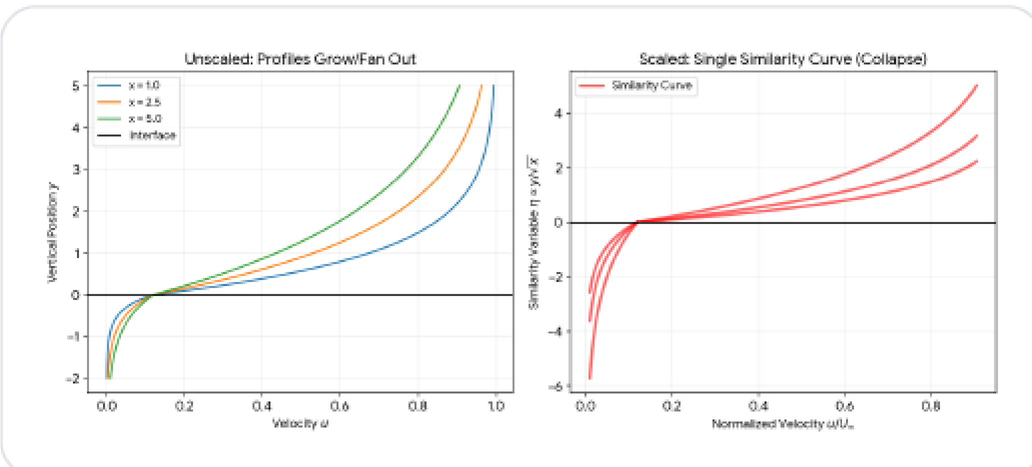
$u_t = \nu u_{yy} \quad u(y,0) = U\,\mathrm{sgn}(y), \quad u(\infty,t) = U, \quad u(-\infty,t) = -U$

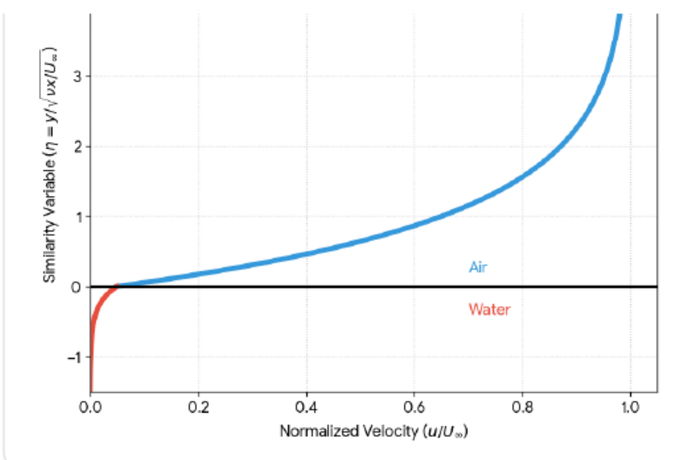
$\delta = \pm 5.52\sqrt{\nu t}$ for $u = \pm 0.95\,U$ S shaped profile but with ±U

(4) Stability: linearized stability parallel stream flow: $Re_{critical} = 0$

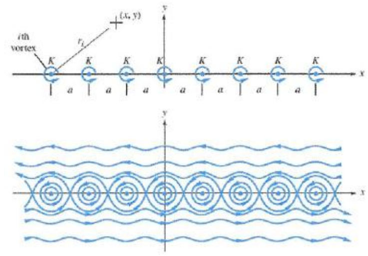
Also related Kelvin–Helmholtz interfacial instability for horizontal interface dividing two ideal fluids with different U and ρ, i.e., ΔU and $\Delta\rho$ jump at interface ⇒ vortex sheet.


Unscaled Velocity Profiles
Height y
Velocity u
x = 1
x = 4
x = 9
Scaled (Similarity) Profile
Similarity Variable η
Normalized Velocity f'
x = 1
x = 4
x = 9

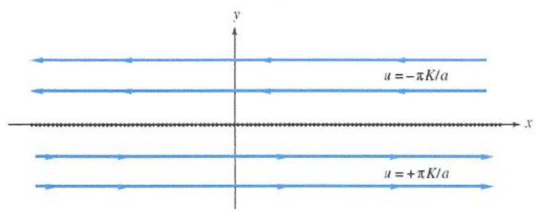

Unscaled: Profiles Grow/Fan Out
Vertical Position y
Velocity u
x = 1.0
x = 2.5
x = 5.0
Interface
Scaled: Single Similarity Curve (Collapse)
Similarity Curve
Normalized Velocity u/U∞


Similarity Variable (η = y/√(νx/U∞))
Air
Water
Normalized Velocity (u/U∞)

Potential flow solution for vortex sheet: Superposition infinite row equally spaced vortices of equal strength.



From afar (i.e. $y \geq a$) looks like a thin sheet with velocity discontinuity.



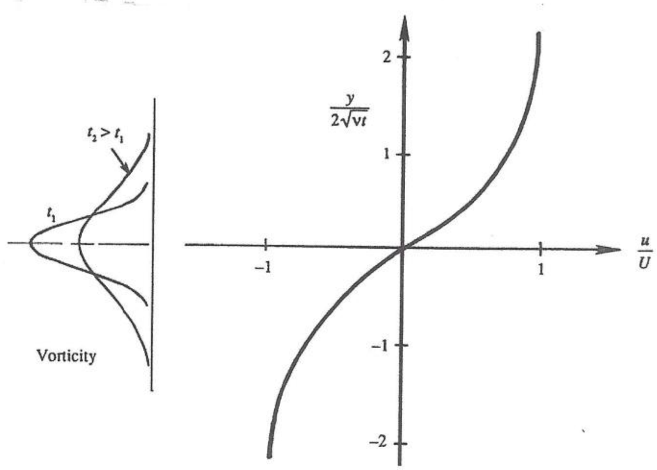


FIGURE 9.14 Viscous thickening of a vortex sheet. The left panel indicates the vorticity distribution at two times, while the right panel shows the velocity field solution in similarity coordinates. The upper half of this flow is equivalent to a temporally developing boundary layer.

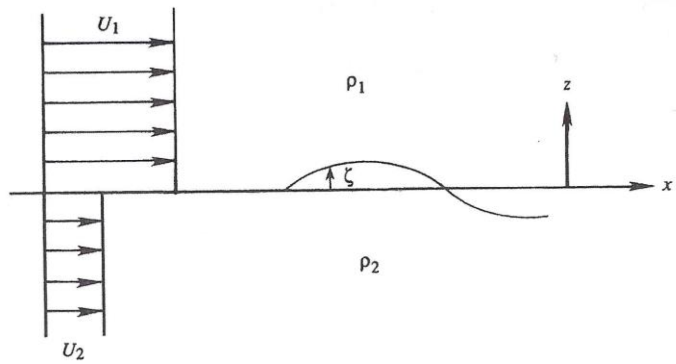


FIGURE 11.2 Basic flow configuration leading to the Kelvin-Helmholtz instability. Here the velocity and density profiles are discontinuous across an interface nominally located at $z = 0$. If the small vertical perturbation $\zeta(x,t)$ to this interface grows, then the flow is unstable.

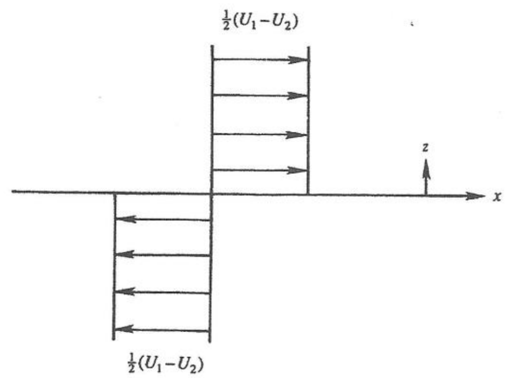


FIGURE 11.3 Background velocity field for the Kelvin-Helmholtz instability as seen by an observer traveling at the average velocity $(U_1 + U_2)/2$ of the two layers. When the densities of the two layers are equal, a disturbance to the interface will be stationary in this frame of reference.