Part 2: Laminar Boundary Layer: $\boldsymbol{Re_{crit}} = \mathbf{91,000}$; $\boldsymbol{Re_{trans}} = \mathbf{5 \times 10^5 - 3 \times 10^6}$.

Similarity solutions (2D, steady, incompressible): method of reducing PDE to ODE by appropriate similarity transformation; also, because of transformation at least one coordinate lacks origin such that the solution collapses to same form at all length or time scales.

$$u_x + v_y = 0$$
$$uu_x + vu_y = UU_x + \nu u_{yy}$$

BCs: $u(x,0) = v(x,0) = 0$ no slip
$u(x,\infty) = U(x)$ matching outer flow

For Similarity $\frac{u(x,y)}{U(x)} = f'\left(\frac{y}{g(x)}\right)$ expect $g(x)$ related to $\delta(x)$
Or in terms of stream function Ψ : $u = \psi_y$ $v = -\psi_x$
For similarity $\psi = U(x)g(x)f(\eta)$ $\eta = y/g(x)$

$$u = \psi_y = Uf' \quad v = -\psi_x = -(U_x g f + U g_x f - U g_x \eta f')$$

BC:

$$u(x,0) = 0 \Rightarrow U(x) f'(0) = 0 \Rightarrow f'(0) = 0$$
$$v(x,0) = 0 \Rightarrow U_x(x)g(x) f(0) + U(x)g_x(x) f(0) - U(x)g_x(x) \times 0 \times f'(0) = 0$$
$$\Rightarrow (U_x(x)g(x) + U(x)g_x(x)) f(0) = 0$$
$$\Rightarrow f(0) = 0$$
$$u(x,\infty) = U(x) \Rightarrow U(x) f'(\infty) = U(x) \Rightarrow f'(\infty) = 1$$

Write boundary layer equations in terms of ψ

$$\psi_y \psi_{yx} - \psi_x \psi_{yy} = UU_x + \nu \psi_{yyy}$$

Substitute

$$\psi_{yy} = Uf''/g$$

$$\psi_{yyy} = Uf'''/g^2$$

$$\psi_{xy} = U_x f' - Uf''\eta g_x / g$$

Assemble them together:

$$(Uf')\left(U_x f' - Uf''\eta \frac{g_x}{g}\right) - \left(U_x g f + U g_x f - U g_x \eta f'\right)\left(Uf''/g\right)$$

$$= UU_x + \nu\left(U f'''/g^2\right)$$

$$UU_x f'^2 - UU_x f f'' - (U^2 g_x/g) f f'' = UU_x + \nu \frac{U}{g^2} f'''$$

$$UU_x f'^2 - \frac{U}{g}(Ug)_x f f'' = UU_x + \nu \frac{U}{g^2} f'''$$

$$f''' + \underbrace{\frac{g}{\nu}(Ug)_x}_{C_1} f f'' + \underbrace{\frac{g^2}{\nu} U_x}_{C_2}\left(1 - f'^2\right) = 0$$

Where for similarity C_1 and C_2 are constant or function η only

- i.e. for a chosen pair of C_1 and C_2→ $U(x)$, $g(x)$ can be found, i.e., potential flow is NOT known a priori.
- Then solution of $f''' + C_1 f f'' + C_2\left(1 - f'^2\right) = 0$ gives $f(\eta)$→ $u(x, y)$, $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_w = \frac{\mu U f''(0)}{g}$, δ, δ^*, θ, H, C_f, C_D

The Blasius Solution for Flat-Plate Flow

U=constant→ $U_x = 0 \rightarrow C_2 = 0$

Then $C_1 = \frac{U}{\nu} g g_x \neq function(x)$

$\frac{d}{dx}(g^2) = \frac{2C_1\nu}{U} \Longrightarrow g(x) = [2C_1\nu x/U]^{1/2}$

Let $C_1 = 1$, then $g(x) = \sqrt{\frac{2\nu x}{U}} \Longrightarrow \eta = y\sqrt{\frac{U}{2\nu x}} = \frac{y}{\sqrt{\frac{2\nu x}{U}}} \propto \frac{y}{\delta}$

Note $\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}}$, i.e., $\delta = \frac{5x}{\sqrt{\frac{Ux}{\nu}}} = 5\sqrt{\frac{\nu x}{U}}$

$$\psi = U[2\nu x/U]^{1/2} f\left(y\sqrt{\frac{U}{2\nu x}}\right) = \sqrt{2\nu U x}\, f(\eta)$$

$$u = \psi_y = Uf'$$

$$v = -\psi_x = Ug_x(\eta f' - f) = [U\nu/2x]^{1/2}(\eta f' - f) = \frac{U(\eta f' - f)}{\sqrt{2Re_x}}$$

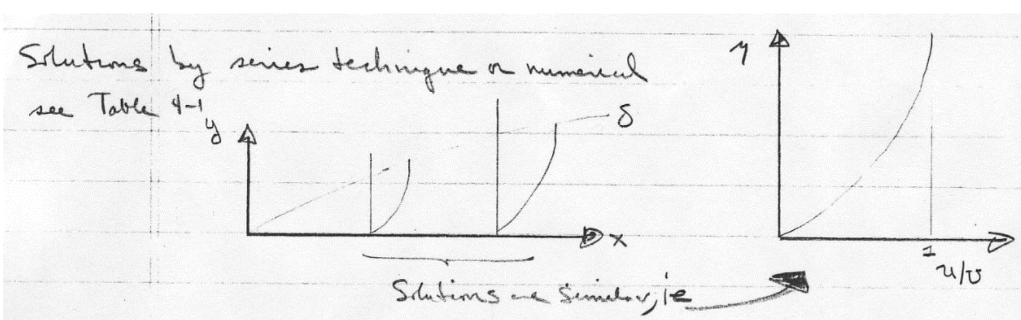
$\Longrightarrow$

$$f''' + ff'' = 0$$

$$f(0) = f'(0) = 0,\ f'(\infty) = 1$$

Blasius Equation for Flat Plate Boundary Layer

Solutions by series or numerical methods



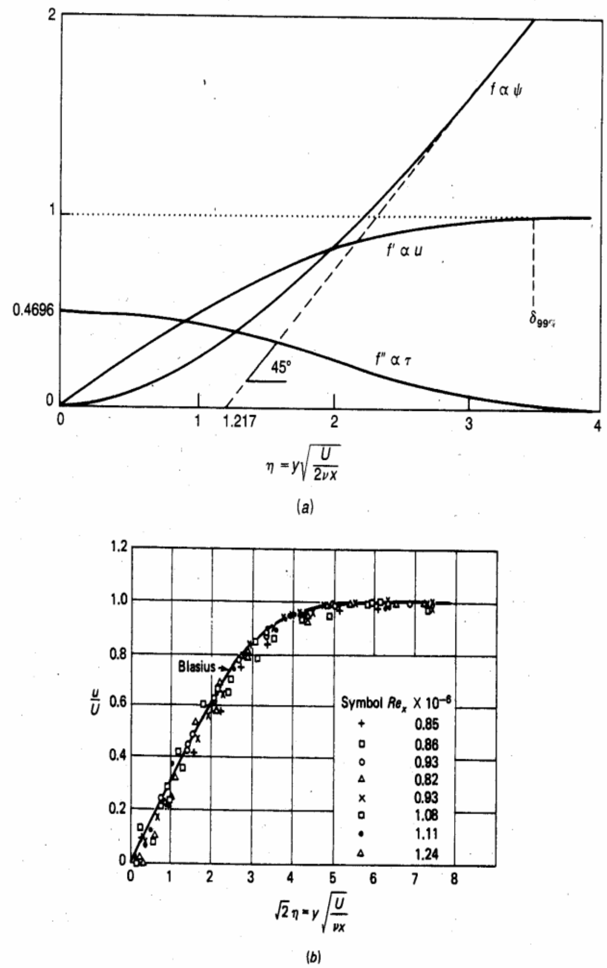


FIGURE 4-6
The Blasius solution for the flat-plate boundary layer: (*a*) numerical solution of Eq. (4-45); (*b*) comparison of $f' = u/U$ with experiments by Liepmann (1943).

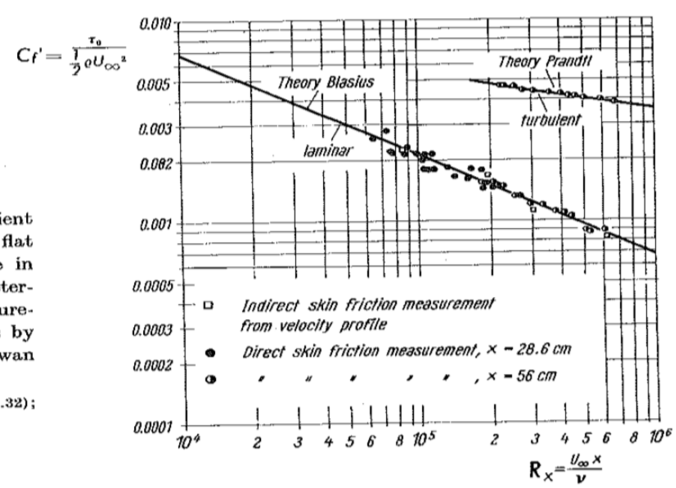


Fig. 7.10. Local coefficient of skin friction on a flat plate at zero incidence in incompressible flow, determined from direct measurement of shearing stress by Liepmann and Dhawan [6, 18]
Theory: laminar from eqn. (7.32); turbulent from eqn. (21.12)

TABLE 4-3

Highly resolved numerical solution of the Blasius equation for flow over a flat plate, Eq. (4-60)

η	$\eta/\eta_{99\%}$	$f(\eta)$	$f'(\eta)$	$f''(\eta)$	$f'''(\eta)$
0.0	0.00000	0.00000	0.00000	0.46960	0.00000
0.0	0.05761	0.00939	0.09391	0.46931	−0.00441
0.4	0.11521	0.03755	0.18761	0.46725	−0.01755
0.6	0.17282	0.08439	0.28058	0.46173	−0.03896
0.8	0.23042	0.14967	0.37196	0.45119	−0.06753
1.0	0.28803	0.23299	0.46063	0.43438	−0.10121
1.2	0.34563	0.33366	0.54525	0.41057	−0.13699
1.4	0.40324	0.45072	0.62439	0.37969	−0.17114
1.6	0.46084	0.58296	0.69670	0.34249	−0.19966
1.8	0.51845	0.72887	0.76106	0.30045	−0.21899
2.0	0.57606	0.88680	0.81669	0.25567	−0.22673
2.2	0.63366	1.05495	0.86330	0.21058	−0.22215
2.4	0.69127	1.23153	0.90107	0.16756	−0.20636
2.6	0.74887	1.41482	0.93060	0.12861	−0.18196
2.8	0.80648	1.60328	0.95288	0.09511	−0.15249
3.0	0.86408	1.79557	0.96905	0.06771	−0.12158
3.2	0.92169	1.99058	0.98036	0.04637	−0.09230
3.3	0.95049	2.08883	0.98456	0.03781	−0.07899
3.4	0.97929	2.18747	0.98797	0.03054	−0.06679
3.47†	1.00000	2.25856	0.99000	0.02603	−0.05878
3.5	1.00810	2.28641	0.99071	0.02441	−0.05582
3.6	1.03690	2.38559	0.99289	0.01933	−0.04611
3.8	1.09451	2.58450	0.99594	0.01176	−0.03039
4.0	1.15211	2.78389	0.99777	0.00687	−0.01914
4.2	1.20972	2.98356	0.99882	0.00386	−0.01152
4.4	1.26732	3.18338	0.99940	0.00208	−0.00663
4.6	1.32493	3.38330	0.99970	0.00108	−0.00366
4.8	1.38253	3.58325	0.99986	0.00054	−0.00193
5.0	1.44014	3.78323	0.99994	0.00026	−0.00098
5.2	1.49774	3.98323	0.99997	0.00012	−0.00047
5.4	1.55535	4.18322	0.99999	0.00005	−0.00022
5.6	1.61296	4.38322	1.00000	0.00002	−0.00010
5.8	1.67056	4.58322	1.00000	0.00001	−0.00004
6.0	1.72817	4.78322	1.00000	0.00000	−0.00002

† Actual value to 16 significant digits: $\eta_\delta = 3.471886880405967$.

$$\frac{u}{U} = 0.99 \text{ when } \eta = 3.5 \rightarrow \frac{\delta}{x} = \frac{5}{\sqrt{Re_x}} \qquad Re_x = \frac{Ux}{\nu}$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \int_0^\infty (1 - f')d\eta \sqrt{\frac{2\nu x}{U}} \rightarrow \boxed{\frac{\delta^*}{x} = \frac{1.7208}{\sqrt{Re_x}}}$$

$$\theta = \int_0^\infty \left(1 - \frac{u}{U}\right) \frac{u}{U} dy = \int_0^\infty (1 - f') f' \sqrt{\frac{2\nu x}{U}} d\eta \rightarrow \boxed{\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}}$$

$$\boxed{\frac{\delta^*}{\theta} = H = shape\ parameter\ 2.59}$$

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_w = \frac{\mu U f''(0)}{\sqrt{\frac{2\nu x}{U}}} \rightarrow \boxed{C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re_x}} = \frac{\theta}{x}}$$

$$\boxed{C_D = \frac{D}{\frac{1}{2}\rho U^2 L} = \int_0^L C_f \frac{dx}{L} = 2C_f(L) = \frac{1.328}{\sqrt{Re_L}}} \qquad Re_L = \frac{UL}{\nu}$$

$$\frac{v}{U} = \frac{\eta f' - f}{\sqrt{2Re_x}} \ll 1 \qquad \text{for} \quad Re_x \gg 1$$

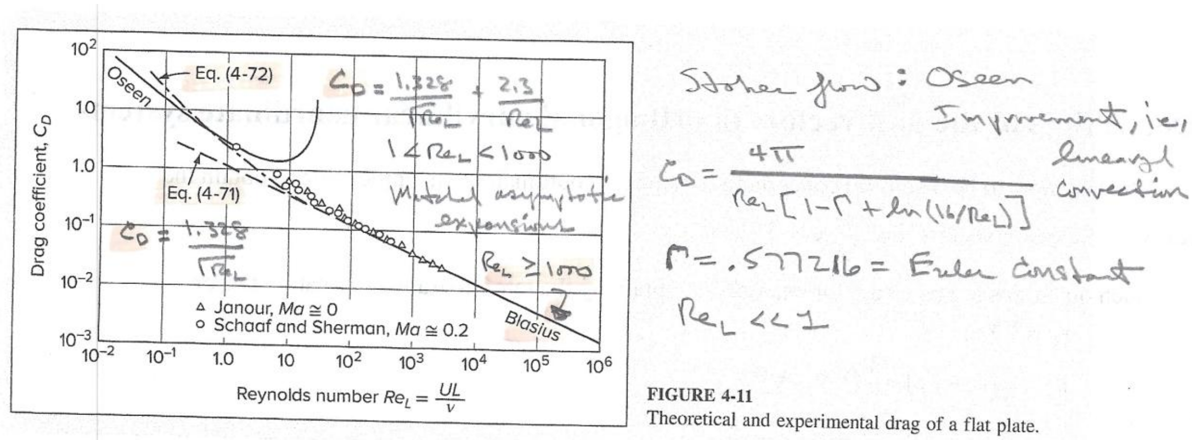
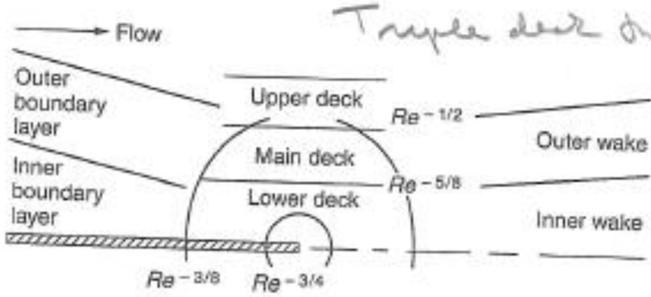


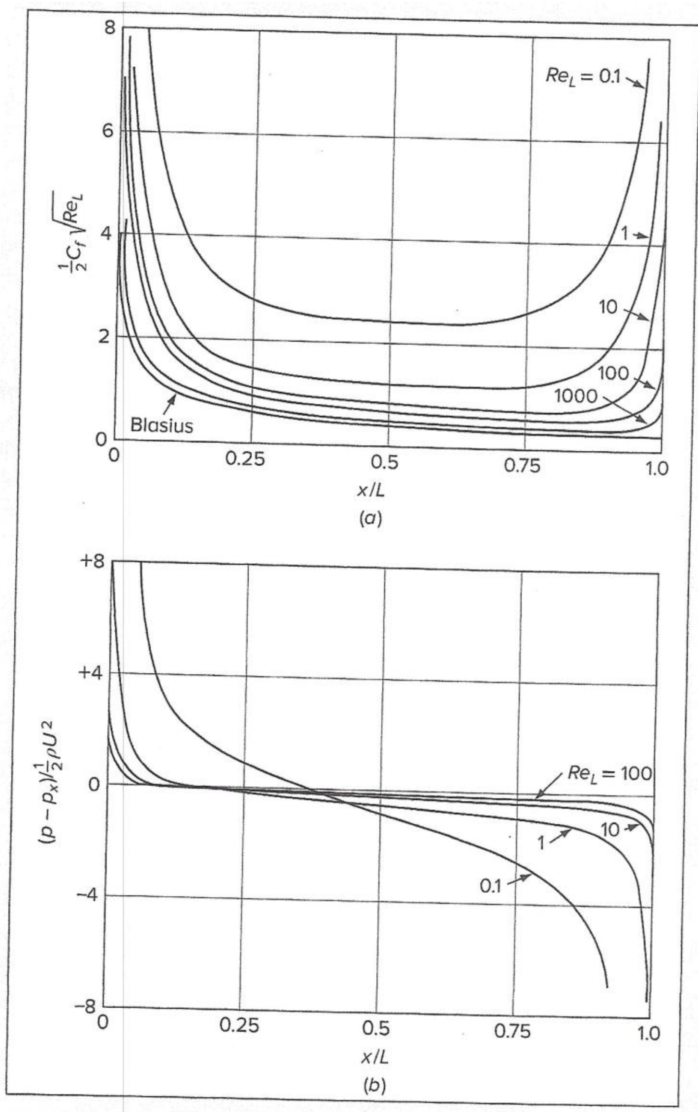
FIGURE 4-11
Theoretical and experimental drag of a flat plate.

Triple deck theory



$$C_D = \frac{1.328}{Re_L^{1/2}} + \frac{2.661}{Re_L^{7/8}} + O(Re_L^{-1})$$

FIGURE 4-40
Sketch of the triple-deck region at the trailing edge of a flat plate, merging into two-layer upstream and downstream regions. [*After Stewartson (1969) and Messiter (1970).*]



NS CFD .1 < Re_L < 1000
low Re large LE & TE effects
BL approximation $Re_L \geq 1000$

FIGURE 4-12
Numerical solution of the full Navier–Stokes equations for flat-plate flow at moderate Reynolds numbers: (*a*) local friction coefficient; (*b*) local surface pressure. [*After Dennis and Dunwoody (1966).*]

Falkner-Skan Wedge Flows

$$f''' + C_1 f f'' + C_2\left(1 - f'^2\right) = 0 \qquad f = f(\eta)$$

$$\eta = y/g(x)$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1 \qquad u/U = f'(\eta)$$

$$C_1 = \frac{g}{v}(Ug)_x \quad C_2 = \frac{g^2}{v}U_x \quad \text{(Blasius Solution: } C_2=0,\ C_1=1)$$

Similarity form of BL equations

Consider $(Ug^2)_x = 2Ugg_x + g^2U_x$

$$= 2Ugg_x + 2g^2U_x - g^2U_x = 2g(Ug_x + gU_x) - g^2U_x$$

$$= 2g(Ug)_x - g^2U_x$$

$$= 2vC_1 - vC_2$$

Hence $(Ug^2)_x = v(2C_1 - C_2)$

Choose $C_1=1$ and let $C_2 = C$

Integrate $Ug^2 = v(2-C)x$ note $g^2 = vC/U_x$

Combine $\frac{UvC}{U_x} = v(2-C)x$

Rearrange $\frac{\frac{dU}{dx}}{U} = \frac{C}{2-C}\frac{1}{x}$

Integrate $ln\,U = \frac{C}{2-C} ln\,x + \ln k$ where $\ln k$ = constant

$$ln\,U = ln\,x^{\frac{C}{2-C}} + ln\,k = \ln kx^{\frac{C}{2-C}}$$

$$U(x) = kx^{C/(2-C)}$$

$$g(x) = \left[\frac{\upsilon C}{U_x}\right]^{\frac{1}{2}} \text{ note } U_x = k\,\frac{C}{2-C}\,x^{(\frac{C}{2-C}-1)} = k\,\frac{C}{2-C}\,x^{(\frac{-2(1-C)}{2-C})}$$

$$g(x) = \left[\frac{\upsilon C}{k\frac{C}{2-C}x^{(\frac{-2(1-C)}{2-C})}}\right]^{\frac{1}{2}} = \left[\frac{\upsilon(2-C)}{k}\,x^{(\frac{2(1-C)}{2-C})}\right]^{\frac{1}{2}} = \sqrt{\frac{\upsilon(2-C)}{k}}\,x^{\frac{1-C}{2-C}}$$

using $a^{1/2}b^{1/2}=(ab)^{1/2}$ and $(a^m)^n=a^{mn}$

Alternatively, $U_x = \frac{C}{2-C}\,\mathrm{k}x^{(C/(2-C))}x^{-1} = \frac{C}{2-C}\,\mathrm{U}x^{-1}$

Such that $g(x) = \left[\frac{\upsilon C}{\frac{C}{2-C}\,\mathrm{U}x^{-1}}\right]^{\frac{1}{2}} = \left[\frac{\upsilon(2-C)x}{\mathrm{U}}\right]^{\frac{1}{2}}$

Change constant: $C = \beta = \frac{2m}{m+1}$ and $m = \frac{\beta}{2-\beta}$

$$U(x) = kx^m$$

$$\eta = \frac{y}{g} = y\sqrt{\frac{m+1}{2}\frac{U}{\nu x}}$$

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0$$

$$f(0) = f'(0) = 0 \text{ and } f'(\infty) = 1$$

Note:

$2\mathrm{m}/(\mathrm{m}+1) = \frac{2\beta}{2-\beta}/(\frac{\beta}{2-\beta} + 1) = \frac{2\beta}{2-\beta}/(\frac{2}{(2-\beta)}) = \beta$

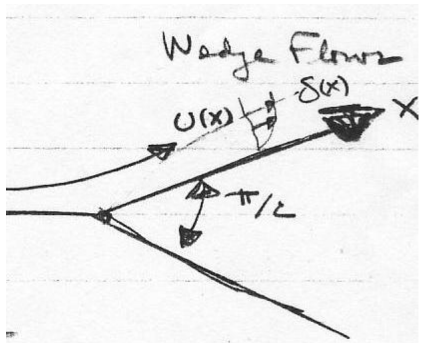
2-C = 2 -2m/(m+1) = 2/(m+1)

$g(x) = \left[\frac{\upsilon(2-C)x}{\mathrm{U}}\right]^{\frac{1}{2}} = \left[\frac{\upsilon 2x}{\mathrm{U(m+1)}}\right]^{\frac{1}{2}}$, i.e., $g(x)^{-1} = \left[\frac{m+1}{2}\frac{U}{\upsilon x}\right]^{\frac{1}{2}}$

Numerical solutions for $-0.19884 \leq \beta \leq 1.0$.

Separation ($\tau_w = 0$)

$$U(x) = kx^m \quad U_x = kx^{m-1} \quad p_x = -\frac{\rho U^2}{x}\frac{\beta}{2-\beta}$$



Solutions show many commonly observed characteristics of BL flow:

- The parameter β is a measure of the pressure gradient, dp/dx. For $\beta > 0$, $dp/dx < 0$and the pressure gradient is favorable. For $\beta < 0$, the $dp/dx > 0$ and the pressure gradient is adverse.

$$\beta = C = C_2 = \frac{g^2}{\upsilon}U_x \quad \eta = y/g(x)$$

- Negative β solutions drop away from Blasius profiles as separation approached.

- Positive β solutions squeeze closer to wall due to flow acceleration.

- Accelerated flow: $\tau_{\max}$ near wall.

- Decelerated flow: $\tau_{\max}$ moves toward $\delta/2$

Inviscid flow past wedge & corner

$U(x) = Kx^m$ exact solution potential flow past wedge or corner shapes

Plane polar coordinates $\nabla^2\psi = 0$, $\nabla \times \underline{u} = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\psi_r) + \frac{1}{r^2}\psi_{\theta\theta} = 0 \qquad v_r = \frac{1}{r}\psi_\theta, \quad v_\theta = -\psi_r$$

$$\psi(r,\theta) = Cr^{m+1}\sin[(m+1)\theta] \qquad \beta = \frac{2m}{m+1} \qquad m = \frac{\beta}{2-\beta}$$

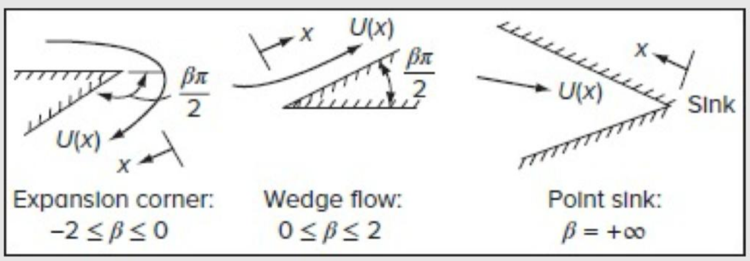
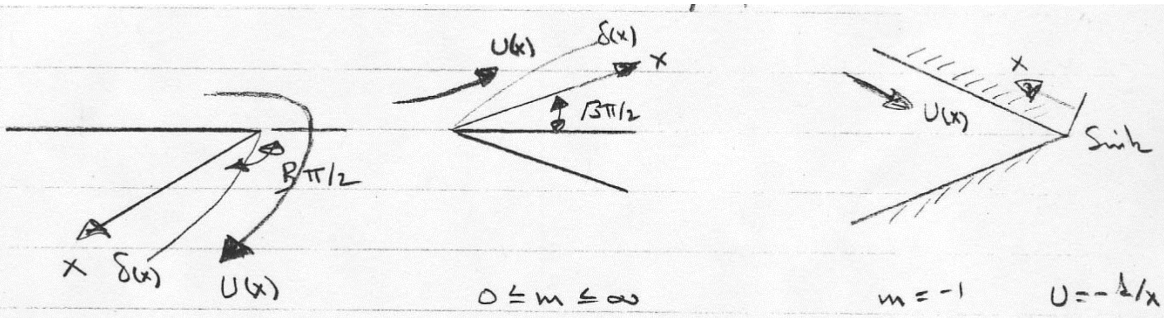


FIGURE 4-15

Some examples of Falkner–Skan potential flows.

$-2 \le \beta \le 0$, $-\frac{1}{2} \le m \le 0$: flow around an expansion corner of turning angle $\beta\pi/2$
$\beta = 0$, $m = 0$: the flat plate
$0 \le \beta \le +2$, $0 \le m \le \infty$: flow against a wedge of half-angle $\beta\pi/2$
$\beta = 1$, $m = 1$: the plane stagnation point (180° wedge)
$\beta = +4$, $m = -2$: doublet flow near a plane wall
$\beta = +5$, $m = -\frac{5}{3}$: doublet flow near a 90° corner
$\beta = +\infty$, $m = -1$: flow toward a point sink [i.e., the boundary-layer version of the Jeffery–Hamel flow in a convergent wedge (Sec. 3-8)].

These are *similar* flows, i.e., for a given β, the velocity profiles all look alike when scaled $U(x)$ and $\delta(x)$. They may also be used, with modest success, to predict the behavior nonsimilar flows.



$-1/2 \le m \le 0$

$-2 \le \beta \le 0$

Expansion Corner

$\beta = -.19884$ (Separation)

$0 \le m \le \infty$

$0 \le \beta \le 2$

Wedge Flow

$m=0$, $\beta=0$ } Flat Plate

$m=1$, $\beta=1$ } Stagnation Point

$m = -1$ $\quad U = -A/x$

$\beta = \infty$ $\quad \eta = \frac{y}{x}\sqrt{\frac{U}{\nu}}$

Sink : Simple Sol involving tanh

Sec. 4.37

BL Sol of Jeffery-Hamel flow

Also, $\beta = 4$, $m = -2$: doublet flow near a plane wall

$\beta = 5$, $m = -5/3$: doublet near a 90° corner

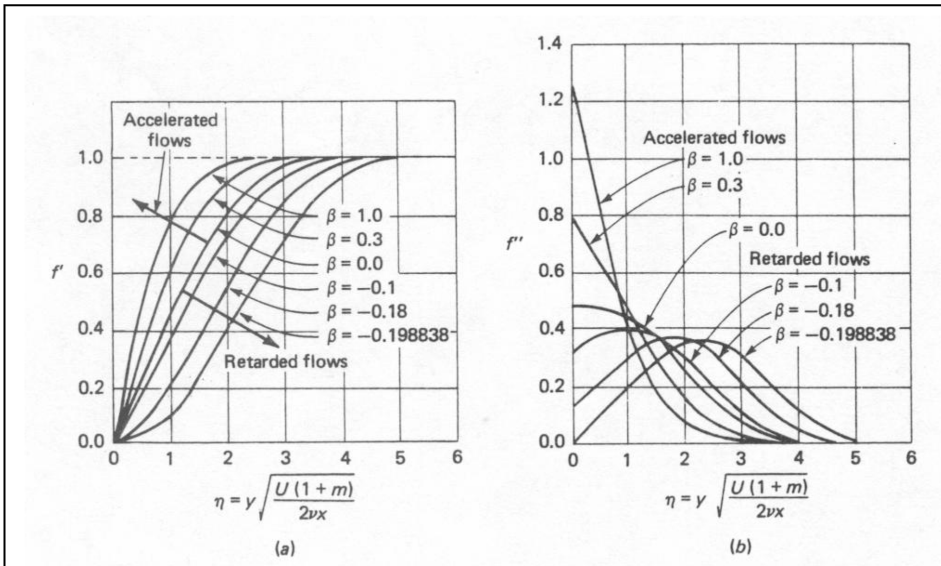


FIGURE 4-11
(a) Velocity profiles and (b) shear-stress profiles for the Falkner-Skan equation.

$u = U(x) f'$ where $U(x) = Kx^m$

↳ velocity at edge of BL

$$\beta = \frac{2m}{m+1} \qquad \eta = \frac{y}{g(x)} = y\left[\frac{m+1}{2}\frac{U(x)}{\nu x}\right]^{1/2} = y\Big/\left[\frac{2}{m+1}\frac{\nu x}{Kx^m}\right]^{1/2}$$

$$\delta(x) = \left[\frac{2\nu}{(m+1)K}\right] x^{(1-m)/2}$$

$$\delta^*(x) = \left(\frac{2}{m+1}\right)^{1/2}\left(\frac{\nu x}{U}\right)^{1/2}\int_0^\infty (1-f')\,d\eta \qquad \left(\frac{2\nu}{(m+1)K}\right) x^{(1-m)/2}$$

$$\tau_w(x) = \mu\left(\frac{m+1}{2}\right)^{1/2}\left(\frac{U^3}{\nu x}\right)^{1/2} f''(0) \qquad \mu\left(\frac{(m+1)K^3}{2\nu}\right)^{1/2} x^{(3m-1)/2}$$

$$\delta \propto x^{(1-m)/2} \quad \delta^* \propto x^{(1-m)/2} \quad \tau_w \propto x^{(3m-1)/2}$$

Special cases:

$U(x) =$ constant 1. Blasius $m = 0, \beta = 0$: $\delta \propto x^{1/2}$, $\delta^* \propto x^{1/2}$,

$\tau_w \propto x^{-1/2}$

$U(x) = kx$ 2. Stagnation Point $m = 1, \beta = 1$:

$\delta,\ \delta^* =$ constant, $\tau_w \propto x$

Linearly increasing velocity U thinning of BL exactly balanced by viscous diffusion such that $\delta,\ \delta^* =$ constant. BL solution = exact solution NS.

$U(x) = kx^3$ 3. $m = 3, \beta = 1.5$: $\delta \propto x^{-1}$, $\delta^* \propto x^{-1}$, $\tau_w =\propto x^4$

The acceleration sets the high value of τ_w, and the viscous diffusion sets the rate at which that stress drops off as you move away from the surface.

Alternative derivation Falkner–Skan equation #1

$u_x + v_y = 0$ i.e. $v_y = -u_x$

Leibniz rule $\int_{a(x)}^{b(x)} \left[\frac{\partial}{\partial x} u(x,y)\right] dy = \frac{d}{dx}\int_a^b u\ dy + u(x,a)\frac{da}{dx} - u(x,b)\frac{db}{dx}$

$$\int_0^v dv = -\int_0^y \frac{\partial u}{\partial x}\, dy = -\frac{\partial}{\partial x}\int_0^y u\ dy - u(x,y)\frac{dy}{dx} + \cancel{u(x,0)}\frac{d0}{dx}$$

=0 since integration across BL for $x =$

$v = -\frac{\partial}{\partial x}\int_0^\delta u\ dy$

x BL equation: $uu_x + vu_y = UU_x + \nu u_{yy}$

$uu_x - \left(\frac{\partial}{\partial x}\int_0^y u\ dy\right) u_y = \text{RHS}$ integro-differential equation in u only

Combine x and y single variable $\eta(x,y)$ to transform PDE to ODE as $f(\eta)$ only to achieve similarity.

Assume: $u(x,y) = U(x)f'(\eta)$ where η dimensionless and $U(x)$ not arbitrary but part of solution.

Let $\eta = yh(x)$ $\quad \eta_x = yh_x = \frac{\eta h_x}{h(x)}$ $\quad \eta_y = h(x)$ $\quad \eta_{yy} = 0$

$u_x = U_x f' + Uf''\eta_x$

$u_y = Uf''\eta_y = Uf''h$

$u_{yy} = Uf'''\eta_y^2 + Uf''\eta_{yy} = Uf'''h^2$

$$Uf'\left(U_x f' + Uf''\frac{\eta h_x}{h}\right) - \left(\frac{\partial}{\partial x}\int_0^y U\,f'\,dy\right)Uf''h = UU_x + \nu Uf'''h^2$$

$$\int_0^y f'\ dy = h^{-1}\int_0^{\eta/h}\frac{df}{d\eta}\,d\eta = \frac{f}{h}$$

$$\frac{\partial}{\partial x}\left(\frac{Uf}{h}\right) = \frac{U_x f}{h} - \frac{Uh_x f}{h^2} + \frac{U}{h}f'\frac{\eta h_x}{h}$$

Since $U(x)$ and $h(x) \neq f(\eta)$ for fixed x

$$Uf'\left(U_x f' + Uf''\frac{\eta h_x}{h}\right) + \left(-\frac{U_x f}{h} + \frac{Ufh_x}{h^2} - \frac{U}{h^2}f'\eta h_x\right)Uf''h$$
$$= UU_x + \nu Uf'''h^2$$

$$UU_x f'^2 + \frac{U^2\eta h_x}{h}f'f'' - UU_x ff'' + \frac{U^2 h_x}{h}ff'' - \frac{U^2\eta h_x}{h}f'f''$$
$$= UU_x + \nu Uh^2 f'''$$

$$UU_x(f'^2 - ff'' - 1) + \frac{U^2 h_x}{h}ff'' = \nu Uh^2 f'''$$

$f''' = \underbrace{\frac{U_x}{\nu h^2}}_{\neq f(x)} (f'^2 - ff'' - 1) + \underbrace{\frac{Uh_x}{\nu h^3}}_{\neq f(x)} ff''$ for similarity only $f(\eta)$

Assume $\eta = C\,y\,x^a$ $\quad U = kx^m \quad m = 2a + 1$

$h = Cx^a \quad U_x = kmx^{m-1} \quad a = \frac{m-1}{2} \quad a - 1 = \frac{m-1}{2} - 1 = \frac{m-3}{2}$

$h_x = aCx^{a-1}$

$$\frac{U_x}{\nu h^2} = \frac{kmx^{m-1}}{\nu c^2 x^{m-1}} = \frac{km}{\nu c^2}$$

$$\frac{Uh_x}{\nu h^3} = \frac{kx^m\left(\frac{m-1}{2}\right)cx^{\frac{m-3}{2}}}{\nu c^3 x^{\frac{3(m-1)}{2}}} = \frac{k\left(\frac{m-1}{2}\right)cx^{\frac{3(m-1)}{2}}}{\nu c^3 x^{\frac{3(m-1)}{2}}} = \frac{k\frac{m-1}{2}}{\nu c^2}$$

$$f''' + \frac{U_x}{\nu h^2}(1 - f'^2) + \underbrace{\left(\frac{U_x}{\nu h^2} - \frac{Uh_x}{\nu h^3}\right)}_{1} ff'' = 0$$

$$\frac{k\left(m - \frac{m-1}{2}\right)}{\nu c^2} = \frac{k\left(\frac{m+1}{2}\right)}{\nu c^2} = 1 \text{ if } c^2 = \frac{k(m+1)}{2\nu}$$

$$\beta = \frac{km}{\nu c^2} = \frac{km}{\frac{\nu k(m+1)}{2\nu}} = \frac{2m}{m+1}$$

$\therefore f''' + \beta(1 - f'^2) + ff'' = 0$

$$\mathrm{h(x)} = [\mathrm{k}\frac{m+1}{2\nu}]^{1/2}x^{\frac{m-1}{2}} = [\mathrm{k}\frac{m+1}{2\nu}x^m x^{m-1}]^{1/2} = [\frac{m+1}{2}\frac{U}{x\nu}]^{1/2} = g(x)^{-1}$$

Alternative derivation Falkner – Skan equation #2: what types of external flows allow similarity solutions BL equations.

Assume self-similar velocity profiles: $u(x,y) = u_e(x)f'(\eta) \quad \eta = \frac{y}{\delta(x)}$

$$\eta = y\delta^{-1} \quad \eta_x = -y\delta^{-2}\delta_x = -\eta\delta^{-1}\delta_x \quad \eta_y = \frac{1}{\delta(x)}$$

u_e, δ determined as part of solution.

$$\underbrace{u = \psi_y \quad v = -\psi_x}_{continuity\ automatically\ satisfied}$$

$$d\psi = \psi_x\, dx + \psi_y\, dy$$

for $x =$ constant $\psi(x,y) - \psi(x,0) = \int_0^y \psi_y\ dy\ = \delta u_e \int_0^\eta \frac{u}{u_e}\, d\left(\frac{y}{\delta}\right) = \delta u_e \int_0^\eta f'\ d\eta = \delta u_e f(\eta)$

$\psi = 0$ at $y\ =\ 0 \ \Rightarrow\ \frac{\psi}{\delta u_e} = f(\eta)$, i.e., $\psi = \delta u_e f(\eta)$ and $\Rightarrow f(0) = 0$

u=$u_e(x)f'(\eta) = \psi_y =\ u_e f' \quad u_x = u_{e_x} f' - u_e f''\eta\delta^{-1}\delta_x$

$$u_y = u_e f''\delta^{-1} \quad u_{yy} = u_e f'''\delta^{-2}$$

$$v = -\psi_x = -f(\delta u_e)_x - (\delta u_e)f'\eta_x = -f\left(\delta_x u_e + \delta u_{e_x}\right) + (\delta u_e)f'\eta\delta^{-1}\delta_x = -f\left(\delta_x u_e + \delta u_{e_x}\right) + \eta u_e f'\delta_x$$

$$uu_x + vu_y = u_e u_{e_x} + \nu u_{yy}$$

$$u_e f'\left(u_{e_x} f' - u_e f''\eta\delta^{-1}\delta_x\right) + \left[-f\left(\delta_x u_e + \delta u_{e_x}\right) + \eta u_e f'\delta_x\right]u_e f''\delta^{-1} = u_e u_{e_x} + \nu u_e f'''\delta^{-2}$$

$$u_e u_{e,x} f'^2 - u_e^2\eta\frac{\delta_x}{\delta}f''f' - \left(\delta_x u_e^2 + \delta u_e u_{e_x}\right)\frac{ff''}{\delta} + u_e^2\eta\frac{\delta_x}{\delta}f'f'' = \text{RHS}$$

$$u_e u_{e_x} f'^2 - \left(\delta_x u_e^2 + \delta u_e u_{e_x}\right)\frac{ff''}{\delta} - u_e u_{e_x} = \frac{\nu u_e}{\delta^2} f'''$$

$$f''' + \frac{u_e u_{e_x}\delta^2}{\nu u_e}(1 - f'^2) + \frac{\delta}{\nu u_e}\left(\delta_x u_e^2 + \delta u_e u_{e_x}\right) ff'' = 0$$

$$f''' + \frac{u_{e_x}\delta^2}{\nu}(1 - f'^2) + \frac{\delta}{\nu}\left(\underbrace{\delta_x u_e + \delta u_{e_x}}_{\frac{d}{dx}(\delta u_e)}\right) ff'' = 0$$

$$f''' + \alpha f f'' + \beta(1 - f'^2) = 0 \qquad \beta = \frac{u_{e_x}\delta^2}{\nu} \quad \alpha = \frac{\delta}{\nu}(\delta u_e)_x$$

ODE for f assuming $\alpha, \beta \neq f(x)$ = constant

note: $2\alpha - \beta = \frac{1}{\nu}\frac{d}{dx}(\delta^2 u_e)$ $\qquad (2\alpha - \beta)(x - x_0) = \frac{1}{\nu}\delta^2 u_e$

Assume $x_0 = 0$ and $\delta(0) = 0$: $\quad \delta = \left[\frac{\nu(2\alpha-\beta)x}{u_e}\right]^{1/2}$

Usually, $\underbrace{u_e, x > 0}_{2\alpha-\beta=1}$; however, may have $\underbrace{\text{apposite sign}}_{2\alpha-\beta=-1}$

$2\alpha - \beta$ any value acceptable. Also, since y/δ similarity variable $y/c\delta$ also OK and c = constant. Thus,

$$\delta = \left[\pm\frac{\nu x}{u_e}\right]^{1/2}$$

Original derivation, whereas sometimes $\alpha = 1$ used such that $\delta = \left[\frac{\nu(2-\beta)x}{u_e}\right]^{1/2}$.

$u_e(x)$ from $\beta = \frac{u_{e_x}\delta^2}{\nu} = \pm \frac{u_{e_x}\nu x}{u_e}/\nu = \pm \frac{x}{u_e}\ u_{e_x} = \mp \frac{x}{\rho u_e^2} p_x$

$\beta = \pm \frac{x}{u_e}\frac{du_e}{dx} \qquad \beta \frac{dx}{x} = \pm \frac{du_e}{u_e}$

$ln\, L^{-1} + \beta\, ln\, x = \pm\, ln\, u_e + ln\, u_0^{-1}$

$\ln\left(\frac{x}{L}\right)^\beta = \pm \ln\left(\frac{u_e}{u_0}\right)$

L, u_0 same sign u_e, x

$m = \beta \quad u_e, x$ same sign

$m = -\beta \quad u_e, x$ opposite sign

$u_e = u_0\left(\frac{x}{L}\right)^m$ For self-similar BL, u_e assumes power law

$$\eta = y/\delta = y\Big/\left[\pm\frac{\nu x}{u_e}\right]^{1/2} = y/L\,/\sqrt{Re}\left(\frac{x}{L}\right)^{\frac{1-m}{2}} \qquad Re = \frac{u_e L}{\nu}$$

For u_e, x same sign $m = \beta \quad 2\alpha - m = 1 \quad \alpha = \frac{m+1}{2}$

$$f''' + \frac{m+1}{2} f f'' + m(1 - f'^2) = 0$$

where for BL flow $u = v = 0 \quad y = 0 \quad u = u_e \quad y \to \infty$

$u(x,y) = u_e(x) f'(\eta) \qquad \eta = \frac{y}{\delta}$

$v(x,y) = -f(u_e\delta)_x + \eta u_e f' \delta_x$

$\therefore f(0) = 0 \quad f'(0) = 0$ and $f'(\infty) = 1$

IC not needed; however, $x_1 = 0 \quad \eta$ singular

FS equation nonlinear ODE $\therefore$ a priori not known which m provides solutions and whether or not unique.

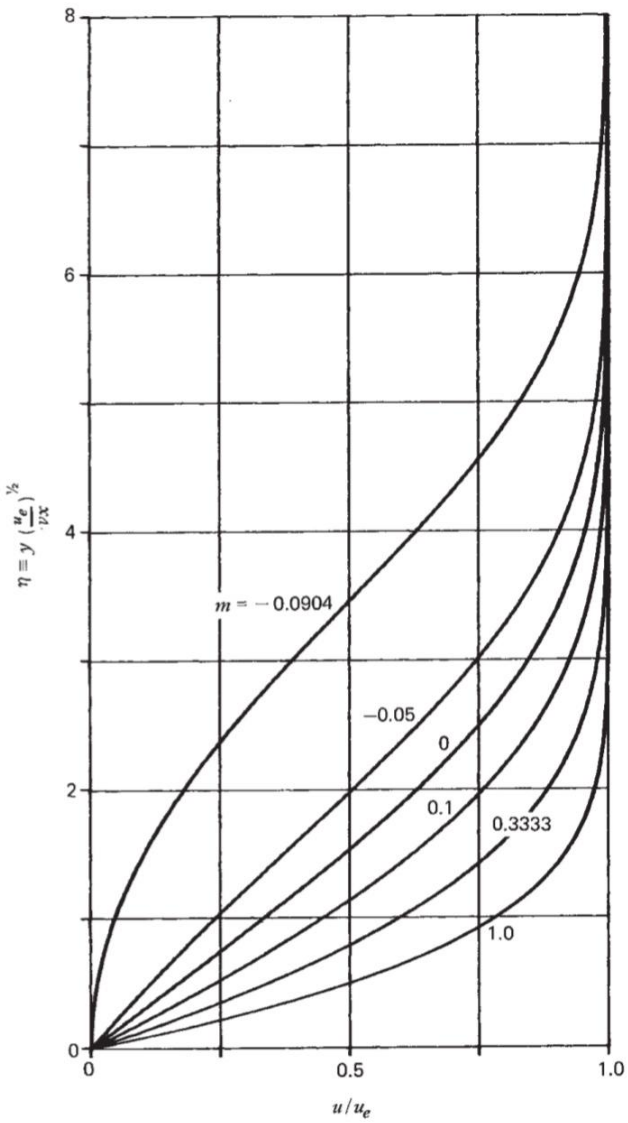


Figure 20.4 Falkner–Skan profiles. The parameter m indicates the external velocity variation through $u_e = u_0 x^m$.

$m \geq 0$ unique solutions for:

$m = 0$:	Blasius flow over a flat plate with a sharp leading edge; also the local flow at any cusp leading edge
$0 < m < 1$:	Flow over a wedge with half-angle $\theta_{1/2} = m/(m+1)$ with $0 < \theta_{1/2} < \pi/2$
$m = 1$:	Hiemenz flow toward a plane stagnation point: Section 11.9
$1 < m < 2$:	Flow into a corner with $\theta_{1/2} > \pi/2$; a flow of this type may be difficult to produce experimentally
$m > 2$:	No corresponding simple ideal flow

Many people have contributed to the classification and computation of Falkner–Skan flows. Rosenhead (1963) contains a good summary by C. W. Jones and E. J. Watson. From these works some of the complicated behavior at negative values of m can be pieced together. When $-0.0904 < m < 0.0$, there are an infinite number of solutions for each value of m (Fig. 20.4). However, not all of these solutions are physically acceptable. One of the main arguments in establishing boundary layer theory is that the viscous effects are confined to a thin region near the wall. In light of this fact, people have proposed that a boundary layer should approach the free stream exponentially:

$$1 - u^* = 1 - f' \sim Ae^{-B\eta} \qquad \text{as} \quad \eta \to \infty$$

If this condition is applied, there are only two known acceptable solutions for each m in the range $-0.0904 < m < 0$. One of the solutions has $u > 0$ for all η, while the other has the interesting characteristic that there is backflow for a small region near the wall [Stewartson's (1954) reverse-flow profiles]. When m is exactly equal to -0.0904, only one solution exists. This profile has zero shear stress at the wall and therefore is on the verge of separating for all x.

For $-1 < m < -0.0904$, all solutions for a given m tend to oscillate about $f' = 1$ as η becomes infinite. At each value of m, one of these solutions has just one region where the velocity $f' > 1$, and then $f \to 1$ exponentially. Because a laminar boundary layer with these super velocities may be difficult to produce experimentally, some workers reject these solutions as physically impossible.

The case $m = -1$ with u_e and x having opposite signs, $u = -a_0/x(u_0 = -a_0)$, represents a solid wall in the flow field of an ideal line sink. When two walls are present, the problem represents the flow into a wedge. The differential equation in this case has an exact closed-form solution (Problem 20.9). On the other hand, the equivalent problem with the sign changed so the flow comes from a source ($m = -1$ and $u = +u_0/x$) has no solution. This means that boundary layer theory does not produce a similarity solution for flows in a flat-wall diffuser. These flows require a nonsimilar solution.

Most of the complicated behavior in Falkner–Skan solutions happens when m is between -1 and 0. When $m < -1$ we again find a unique solution. All solutions in the range $m < -1$ have the flow going from large x toward $x = 0$. Hence, the flows are strongly accelerated with $u = -a_0(x/L)^m$, $m < -1$.

Vertical velocity

From outside BL, as $Re \uparrow \infty$ $v/u_0 \to 0$ since $\delta \to 0$

From inside BL, vu_y same order remaining terms x-momentum equation since $u_y \to \infty$ and $v \to 0$ as $Re \to \infty$, $\text{i.e.}, 0 \times \infty$ finite.

$$v^* = \frac{v}{u_0}\left(\frac{L}{\delta}\right) = \frac{v}{u_0}\sqrt{Re}$$

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}}$$

$$u\,u_x \sim \frac{U^2}{L} \propto \nu\,u_{yy} \sim \frac{\nu U}{\delta^2}$$

$$\delta^2 \sim \frac{\nu L}{U}\ , \ \text{i.e.}, \ \frac{\delta}{L} \sim \frac{1}{\sqrt{Re}}$$

(1) Inviscid flow near wall continuity equation

$$\frac{du_e}{dx} + \frac{dv}{dy} = 0 \Rightarrow v|_{inviscid} = \int_0^y \frac{\partial v}{\partial y}\,dy = -\frac{du_e}{dx}\int_0^y dy$$

Assuming $\frac{du_e}{dx}$ constant $v|_{inviscid} \sim -\frac{du_e}{dx}y$ as $y \to 0$ and $\delta = 0$.

Inviscid flow near wall

1. $u_e = u_e(x)$

2. $v(x, 0) = 0$ and $\frac{\partial v}{\partial n} = 0$

3. $v \sim Ay$, i.e., grows linearly

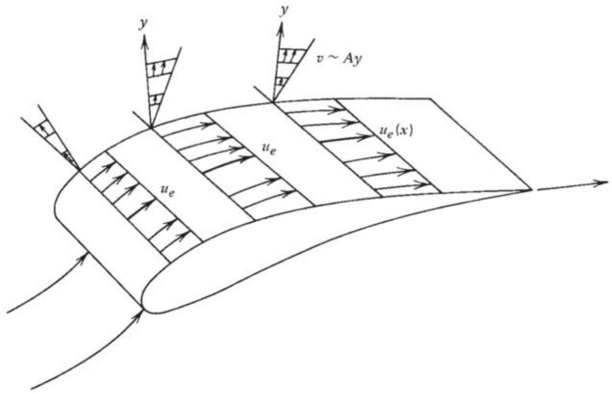


Figure 20.7 View from outside the boundary layer looking at the behavior of u and v in the inviscid flow at the top of the boundary layer.

(2) Viscous BL flow near wall continuity equation

$$v = \int_0^y \frac{\partial v}{\partial y}\, dy = -\int_0^y \frac{\partial u}{\partial x}\, dy = \frac{d}{dx}\int_0^y (u_e - u)\; dy - \frac{du_e}{dx} y$$

$$v(y \to \infty) = -\underbrace{\frac{du_e}{dx} y}_{inviscid\ v} + \underbrace{\frac{d}{dx}(u_e \delta^*)} \text{ where } \delta^* = \int_0^\infty \left(1 - \frac{u}{u_e}\right) dy$$

influence BL profile on v,
i.e., v_{BL} correction inviscid flow

2^{nd}order solution inviscid solution instead of $v(wall) = 0$ use $v(wall) = \frac{d}{dx}(u_e \delta^*)$

$v(\eta)$ for Falkner–Skan BL solutions various m

$$v = -\psi_x = -f \frac{d}{dx}(u_e \delta) + \eta u_e f' \frac{d\delta}{dx}$$

$$= -\sqrt{\frac{(m+1)\nu U(x)}{2x}}\left[f(\eta) + \frac{m-1}{m+1}\eta f'(\eta)\right]$$

$$\beta = \frac{2m}{m+1}$$

For m > 0 favorable pressure gradient and v < 0, i.e., entrainment, whereas m < 0 adverse pressure gradient and v > 0, i.e., expulsion.

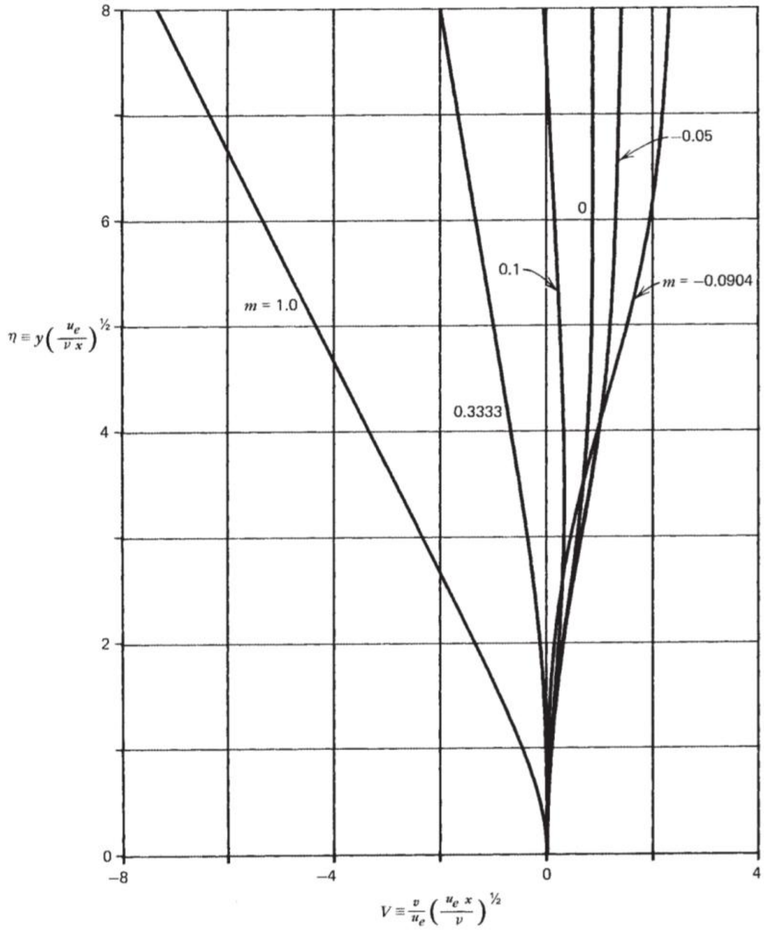


Figure 20.8 Vertical-velocity profiles for Falkner–Skan boundary layers.

m	β
-0.0904	-0.199
-0.05	-0.105
0	0
0.1	0.182
0.3333	0.5
1	1

For η → ∞ approaches nondimensional form previous equation:

$$V(\infty) = \frac{v}{u_e}\left(\frac{u_e x}{\nu}\right)^{1/2} = -\underbrace{\beta}_{\Uparrow}\eta + \underbrace{\frac{x}{u_e\delta}\frac{d}{dx}(u_e\delta^*)}_{v\ intercept} \qquad \delta = \left(\pm\frac{\nu x}{u_e}\right)^{1/2}$$

$$\beta \neq 0$$
$$V(\infty) \to \infty$$

Linear f(η) for large η with slope $-\beta$ and intercept $\frac{x}{u_e\delta}\frac{d}{dx}(u_e\delta^*)$. Intercept is the value where the asymptotic linear portion of V(η) crosses the $\eta = 0$ axis. Physically it represents the effect of δ^* on the outer inviscid flow.

Flat Plate with Wall Suction or Blowing

Blowing solution with $v_w \ll U \neq 0$ and $u_w = 0$. Similarity requires

$$v_w = -\psi_x = \sqrt{\frac{\nu U}{2x}}(\eta f' - f)|_{\eta=0}, \qquad \eta = y\sqrt{\frac{U}{2\nu x}}$$

$$= \sqrt{\frac{\nu U}{2x}}\left(-f(0)\right) \qquad \therefore f(0) \neq 0 \text{ sets } v_w \propto x^{-1/2}$$

Same Blasius equation with: $f'(0) = 0$, $f'(\infty) = 1$ and $f(0) \neq 0$

Suction–blowing parameter $v_w^* = \frac{v_w}{U}\sqrt{Re_x} = -\frac{f(0)}{\sqrt{2}}$

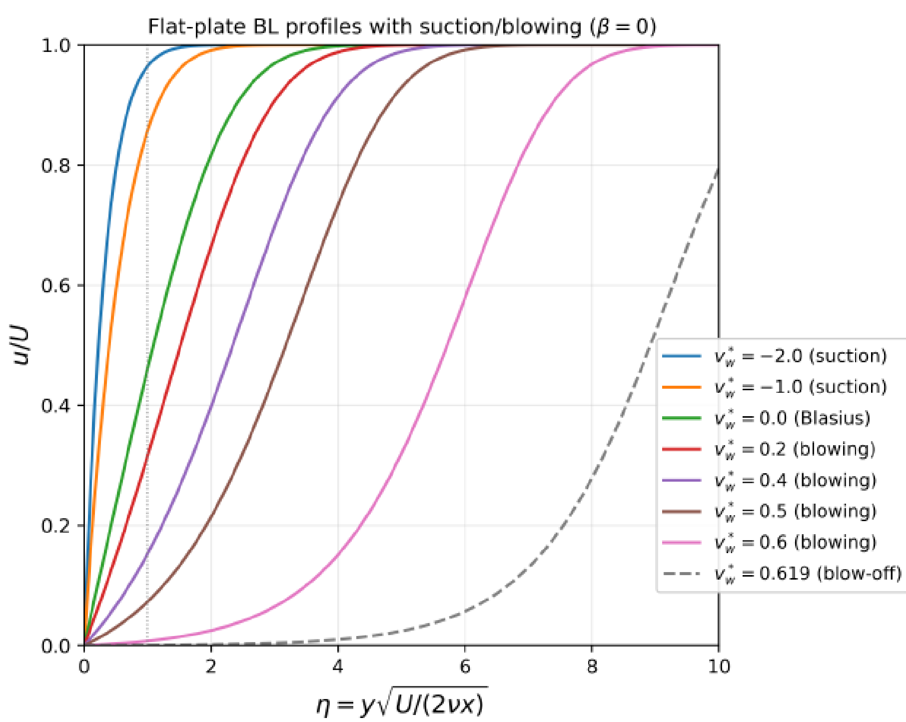


Figure 1: Flat-plate BL velocity profiles $u/U = f'(\eta)$ for suction ($v_w^* < 0$), impermeable Blasius ($v_w^* = 0$), and blowing ($v_w^* > 0$). Dashed curve: blow-off limit $v_w^* = 0.619$ at which $f''(0) = 0$.

$v_w \leq 0$ suction thins BL, increases τ_w similar favorable p_x and stable transition

$v_w > 0$ blowing thickens BL, reduces τ_w similar adverse p_x S-shaped with inflection point, i.e., unstable to turbulent transition

$$v_w^* = 0.619 \Rightarrow \frac{\partial u}{\partial y} = 0 \text{ at } y = 0 \text{ and } u = 0 \quad y \geq 0$$

BL blown off, i.e., $\tau_w = 0$ and BL approximation not valid!

Unlike the exact (uniform-v) solution, the similarity solution is well-posed for blowing because:

- The wall-normal velocity is *non-uniform*: $v_w \propto x^{-1/2}$ decays downstream, so there is no global incompatibility between the no-slip condition and the far-field velocity.
- Both u and v vary with x *and* y through η, eliminating the constraint $u = u(y)$ that caused the breakdown in Section 2.
- The BL approximation requires $v_w \ll U$, i.e. blowing is treated as a perturbation, not as an $O(U)$ forcing.

The blow-off limit ($v_w^* = 0.619$) is nonetheless the BL's analogue of the exact solution's "no solution": beyond it, steady laminar flow cannot be maintained.

See Appendices A and B