Part 3: Momentum Integral Equation

Historically similarity and other AFD methods used for idealized flows and momentum integral methods for practical applications, including pressure gradients, but failure 3D methods motivated 3D BL theory which quickly progressed to modern day CFD.

Momentum integral equation, which is valid for both laminar and turbulent flow:

$$\int_{y=0}^{\infty} (\text{steady flow BL equation } + (u-U)\text{continuity})\, dy$$

$$\underbrace{\frac{\tau_w}{\rho U^2} = \frac{1}{2}C_f = \frac{d\theta}{dx}}_{\text{For flat plate equation} \rightarrow \frac{dU}{dx}=0} + (2+H)\frac{\theta}{U}\frac{dU}{dx}$$

For flat plate equation → $\frac{dU}{dx} = 0$

$$\theta = \int_0^{\delta} \frac{u}{U}\left(1-\frac{u}{U}\right) dy$$

$$\delta^* = \int_0^{\delta} \left(1-\frac{u}{U}\right) dy$$

$$H = \frac{\delta^*}{\theta}$$

The pressure gradient evaluated form the outer potential flow using Bernoulli equation.

$$p + \frac{1}{2}\rho U^2 = \text{constant}$$

$$p_x + \frac{1}{2}\rho 2UU_x = 0$$

$$-p_x/\rho = UU_x$$

Momentum: $uu_x + vu_y = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$ where $\tau = \mu \frac{\partial u}{\partial y}$

$$(u-U)\underbrace{\left(u_x + v_y\right)}_{Continuity} = uu_x + uv_y - Uu_x - Uv_y$$

$$\underbrace{uu_x + vu_y - UU_x - \frac{1}{\rho}\tau_y}_{0} + \underbrace{uu_x + uv_y - Uu_x - Uv_y}_{0} = 0$$

$$-\frac{1}{\rho}\tau_y = -2uu_x - vu_y + UU_x - uv_y + Uu_x + Uv_y$$

$$= \frac{\partial}{\partial x}(uU - u^2) + (U-u)U_x + \frac{\partial}{\partial y}(vU - vu)$$

$$\int_0^\infty -\frac{1}{\rho}\tau_y dy = -\frac{\overset{0}{\cancel{\tau_\infty}} - \tau_w}{\rho} = \frac{\partial}{\partial x}\int_0^\infty u(U-u)dy + U_x\int_0^\infty (U-u)dy + \overset{0}{\cancel{(vU - vu)_0^\infty}}$$

$$\frac{\tau_w}{\rho} = \frac{\partial}{\partial x}\left[U^2\int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right] + U_x U\int_0^\infty \left(1-\frac{u}{U}\right)dy$$

$$= U^2\theta_x + 2UU_x\theta + UU_x\delta^*$$

$$\frac{\tau_w}{\rho U^2} = \frac{1}{2}C_f = \theta_x + (2\theta + \delta^*)\frac{1}{U}\frac{dU}{dx}$$

$$\frac{C_f}{2} = \frac{d\theta}{dx} + (2+H)\frac{\theta}{U}U_x$$

Methods Solution Momentum Integral Equation

Historically two approaches:

(1) One parameter velocity profile
(2) Empirical correlations: Thwaites method

(1) Karman – Pohlhausen Method

① $u(x,y) = U(x) f\left(\frac{y}{\delta}, \Lambda(x)\right)$ guess form velocity profile

$$\frac{u}{U} = 2\eta - 2\eta^3 + \eta^4 + \frac{\Lambda}{6}[\eta(1-\eta)^3]$$

$\eta = \frac{y}{\delta}$ $\quad \Lambda = \frac{\delta^2}{\nu} U_x$ = Pohlhausen parameter

② uses 5 BC for u/U

③ compute: $\theta, \delta^*, H, \tau_w$

④ substitute momentum integral equation for 1st order ODE $\delta(x)$

⑤ with $\delta(x)$ known all variables also known

(2) Accuracy not as good Thwaites method

Issues:

Recall using quadratic guessed profile flat plate velocity profile only ~10% accuracy

Accuracy depends on type of guessed profile.

TABLE 4-1

Boundary-layer predictions from five piecewise analytic profiles with their errors relative to the classic Blasius values

	$\frac{u}{U} = F(\xi)$	$\eta^* = \frac{\delta^*}{\delta}$	$\theta^* = \frac{\theta}{\delta}$	$H = \frac{\delta^*}{\theta}$	$\frac{\delta}{x}\sqrt{Re_x}$	$C_f\sqrt{Re_x}$	$\frac{\delta^*}{x}\sqrt{Re_x}$	L_2 error
1	$2\xi - \xi^2$	0.333 3.1%	0.133 0.25%	2.500 3.5%	5.477 9.5%	0.730 10%	1.826 6.1%	0.020
2	$\frac{3}{2}\xi - \frac{1}{2}\xi^3$	0.375 9.0%	0.139 4.7%	2.692 4.0%	4.641 7.2%	0.646 2.6%	1.740 1.1%	0.034
3	$2\xi - 2\xi^3 + \xi^4$	0.300 13%	0.118 12%	2.554 1.4%	5.836 17%	0.685 3.2%	1.751 1.8%	0.054
4	$\sin\left(\frac{1}{2}\pi\xi\right)$	0.363 5.6%	0.137 2.7%	2.660 2.7%	4.795 4.1%	0.655 1.3%	1.743 1.3%	0.021
5	$\frac{5}{3}\xi - \xi^3 + \frac{1}{3}\xi^4$	0.350 1.7%	0.134 0.52%	2.618 1.1%	4.993 0.13%	0.668 0.53%	1.748 1.6%	0.008
	Blasius (1908)	0.344	0.133	2.59	5	0.664	1.72	n/a

$$L_2 \text{ error } = \left[\int_0^1 (F - F_{Blasius})^2 \; d\eta\right]$$

1, 2, 3 Pohlhausen (1921)
4 Schlichting (1979)
5 Majdalani & Xuan (2020)

Pohlhausen paradox: increasing order, i.e., # BC profile can satisfy does not improve accuracy

BC: $u(x, 0) = 0$ — 1. no slip
$u(x, \delta) = U$ — 2. matching
$u_y(x, \delta) = 0$ — 3. smooth merge U
$\mu u_{yy}(x, 0) = p_x$ — 4. correct balance momentum $y = 0$
$u_{yy}(x, \delta) = 0$ — 5. $\mu \frac{\partial}{\partial y}\tau_y = 0$ zero shear stress at δ

However, the Blasius profile does not in fact satisfy all these conditions!

Note 1 satisfies BCs 1–3 and 2–5 satisfy BCs 1–4 but only 3 satisfy BCs 1–5

↑ largest L_2 error

BC 5 true at $y \to \infty$ but not true $y = \delta$ where $u_{yy}|_{Blasius} = -0.7085$

Other differences such as initial slope $F'(0)$ and $F(1)$ and $F'(1)$

TABLE 4-2

Endpoint properties of the piecewise analytic velocity profiles and their corresponding Blasius values

$\frac{u}{U} = F(\xi)$	$F(0)$	$F'(0)$	$F''(0)$	$F(1)$	$F'(1)$	$F''(1)$
$2\xi - \xi^2$	0	2.000	−2	1.000	0	−2.000
$\frac{3}{2}\xi - \frac{1}{2}\xi^3$	0	1.500	0	1.000	0	−3.000
$2\xi - 2\xi^3 + \xi^4$	0	2.000	0	1.000	0	0.000
$\sin(\frac{1}{2}\pi\xi)$	0	1.571	0	1.000	0	−2.467
$\frac{5}{3}\xi - \xi^3 + \frac{1}{3}\xi^4$	0	1.667	0	1.000	0	−2.000
Blasius (1908)	0	1.630	0	0.990	0.0904	−0.709

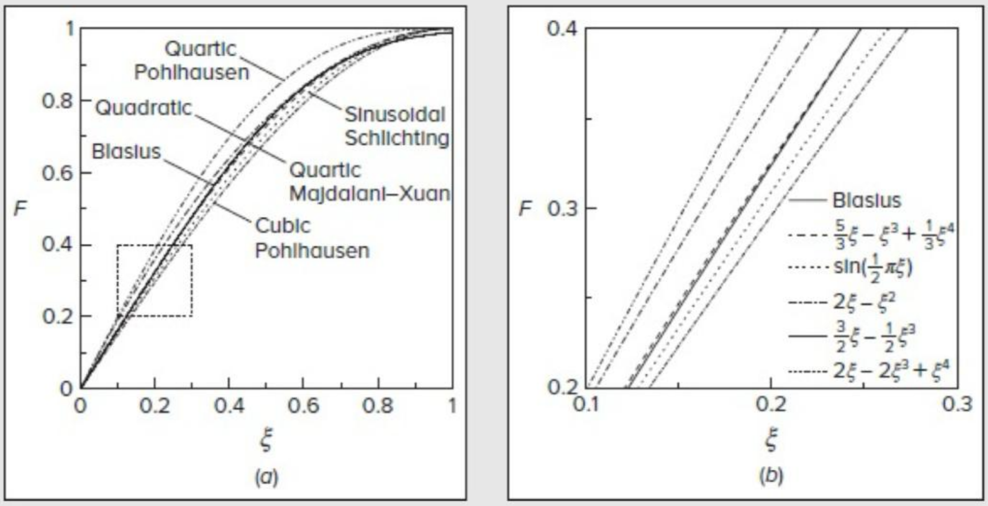


FIGURE 4-5

Comparison of five analytic approximations to the Blasius solution (solid line) including Pohlhausen's quadratic, cubic, and quartic polynomials (chained lines) as well as Schlichting's sinusoidal (dotted) and Majdalani–Xuan's quartic (dashed) profiles across (*a*) the boundary layer and (*b*) a designated quadrant where individual deviations from the Blasius curve are magnified.

Presumably these differences compounded for $p_x \neq 0$

Thwaites Method (1949)

Pressure gradient parameter $\lambda = \frac{\theta^2}{\nu}\frac{dU}{dx} = (\frac{\theta}{\delta})^2\Lambda$

where $\Lambda = \frac{\delta^2}{\nu}\frac{dU}{dx} = -p_x\frac{\delta^2}{\mu U}$ is the Pohlhausen parameter.

Multiply momentum integral equation by $\frac{U\theta}{\nu}$

$$\frac{\tau_w\theta}{\mu U} = \frac{U\theta}{\nu}\frac{d\theta}{dx} + \frac{\theta^2}{\nu}\frac{dU}{dx}(2 + H)$$

The equation is dimensionless and, LHS and H can be correlated with λ as shear and shape-factor correlations:

$$\frac{\tau_w\theta}{\mu U} = S(\lambda) = (\lambda + 0.09)^{0.62}$$

$$H = \delta^*/\theta = H(\lambda) = \sum_{i=0}^{5} a_i(0.25 - \lambda)^i$$

a_i = (2, 4.14, -83.5, 854, -3337, 4576)

Note

$$\frac{U\theta}{\nu}\frac{d\theta}{dx} = \frac{1}{2}U\frac{d}{dx}\left(\frac{\theta^2}{\nu}\right)$$

Substitute above into momentum integral equation.

$$S(\lambda) = \frac{1}{2}U\frac{d}{dx}\left(\frac{\theta^2}{\nu}\right) + \lambda(2 + H)$$

$$U\frac{d(\lambda/U_x)}{dx} = 2[S - \lambda(2 + H)] = F(\lambda)$$

$F(\lambda) = 0.45 - 6\lambda$ based on AFD and EFD

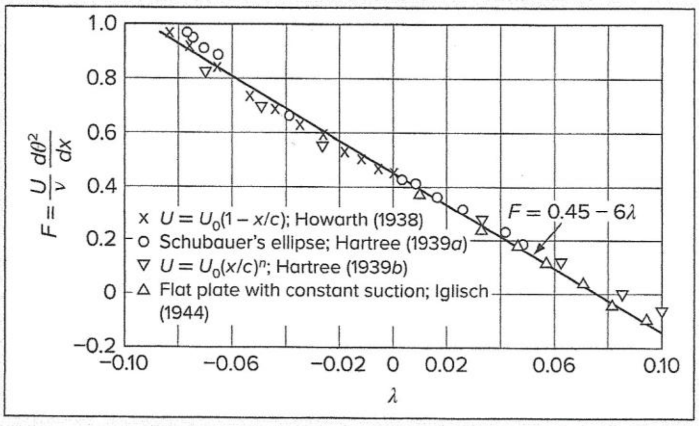


FIGURE 4-27
Empirical correlation of the boundary-layer function in Eq. (4-156). [*After Thwaites* (*1949*).]

Define $z = \frac{\theta^2}{\nu}$ so that $\lambda = z\frac{dU}{dx}$

$$U\frac{dz}{dx} = 0.45 - 6\lambda = 0.45 - 6z\frac{dU}{dx}$$

$$U\frac{dz}{dx} + 6z\frac{dU}{dx} = 0.45$$

$$\frac{1}{U^5}\frac{d}{dx}(zU^6) = U\frac{dz}{dx} + 6z\frac{dU}{dx} = 0.45$$

$$d(zU^6) = 0.45U^5 dx$$

$$zU^6 = 0.45\int_0^x U^5 dx + C$$

$$\Rightarrow \theta^2 = \theta_0^2 + \frac{0.45\nu}{U^6}\int_0^x U^5 dx$$

$\theta_0(x=0) = 0$ and U(x) known from potential flow solution.

Complete solution:

$$\lambda = \lambda(\theta) = \frac{\theta^2}{\nu}\frac{dU}{dx}$$

$$\frac{\tau_w \theta}{\mu U} = S(\lambda)$$

$$\delta^* = \theta H(\lambda)$$

Accuracy: mild p_x ± 5% and strong adverse p_x (τ_w near 0) ± 15%

TABLE 4-8

Shear and shape functions correlated by Thwaites (1949)

λ	$H(\lambda)$	$S(\lambda)$	λ	$H(\lambda)$	$S(\lambda)$
+0.25	2.00	0.500	–0.056	2.94	0.122
0.20	2.07	0.463	–0.060	2.99	0.113
0.14	2.18	0.404	–0.064	3.04	0.104
0.12	2.23	0.382	–0.068	3.09	0.095
0.10	2.28	0.359	–0.072	3.15	0.085
+0.080	2.34	0.333	–0.076	3.22	0.072
0.064	2.39	0.313	–0.080	3.30	0.056
0.048	2.44	0.291	–0.084	3.39	0.038
0.032	2.49	0.268	–0.086	3.44	0.027
0.016	2.55	0.244	–0.088	3.49	0.015
0.0	2.61	0.220	–0.090	3.55 (Separation)	0.000
–0.016	2.67	0.195			
–0.032	2.75	0.168			
–0.040	2.81	0.153			
–0.048	2.87	0.138			
–0.052	2.90	0.130			

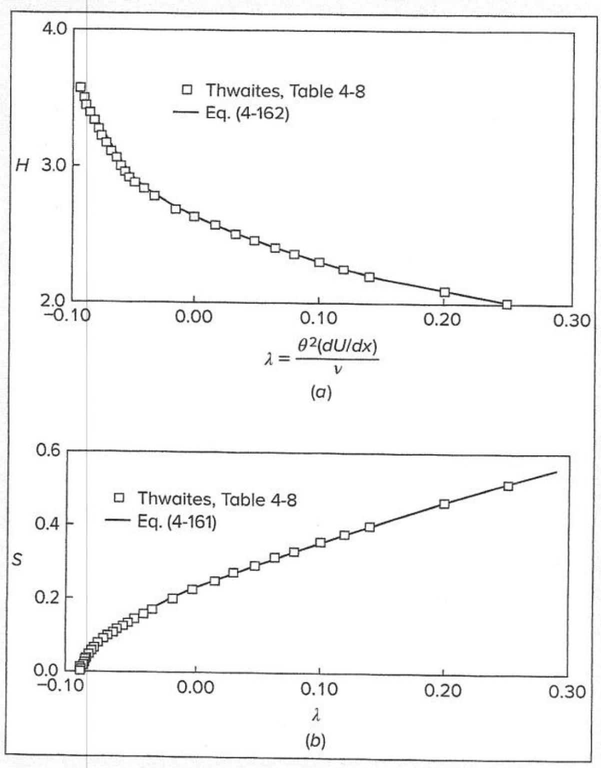


FIGURE 4-28
The laminar boundary-layer correlated functions by Thwaites (1949): (*a*) shape factor; (*b*) shear stress with curve fits.

Separation predicted within 4%; however, large scale separation causes viscous/inviscid interaction and alters imposed external U(x) and $p_x(x)$

TABLE 4-9

Laminar-separation-point prediction by Thwaites' method

$U(x)$	x_{sep} (exact)	Thwaites x_{sep}	Thwaites Error [%]
Howarth (1938)			
$1 - x$	0.120	0.123	+2.5
Tani (1949)			
$1 - x^2$	0.271	0.268	−1.1
$1 - x^4$	0.462	0.449	−2.8
$1 - x^8$	0.640	0.621	−3.0
Terrill (1960)			
$\sin(x)$	1.823	1.800	−1.3
Curle (1958)			
$x - x^3$	0.655	0.648	−1.1
Görtler (1957)			
$\cos(x)$	0.389	0.384	−1.3
$(1 - x)^{1/2}$	0.218	0.221	+1.3
$(1 - x)^2$	0.0637	0.0652	+2.4
$(1 + x)^{-1}$	0.151	0.158	+4.6
$(1 + x)^{-2}$	0.0713	0.0739	+3.6

Pohlhausen Velocity Profile:

$\frac{u}{U} = f(\eta) = a\eta + b\eta^2 + c\eta^3 + d\eta^4$ with $\eta = \frac{y}{\delta}$

a, b, c, d determined from boundary conditions:

1) $y = 0 \rightarrow u = 0,\ u_{yy} = -\frac{U}{\nu}U_x$
2) $y = \delta \rightarrow u = U,\ u_y = 0,\ u_{yy} = 0$

$\rightarrow \frac{u}{U} = F(\eta) + \Lambda G(\eta),\ -12 \leq \Lambda \leq 12 \qquad \Lambda = \frac{\delta^2}{\nu}\frac{dU}{dx} = -p_x\frac{\delta^2}{\mu U}$

separation (experiment: $\Lambda_{separation} = -5$)

$F(\eta) = 2\eta - 2\eta^3 + \eta^4$

$G(\eta) = \frac{\eta}{6}(1-\eta)^3$

$\lambda = \lambda(\Lambda) = \left(\frac{37}{315} - \frac{\Lambda}{945} + \frac{\Lambda^2}{9072}\right)\Lambda$

Profiles are realistic, except near separation. In guessed profile methods u/U directly used to solve momentum integral equation numerically, but accuracy not as good as empirical correlation methods; therefore, use Thwaites method to get λ, etc., and then use λ to get Λ and plot u/U.

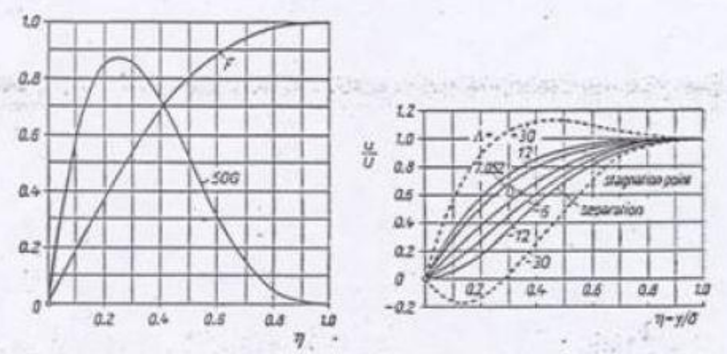
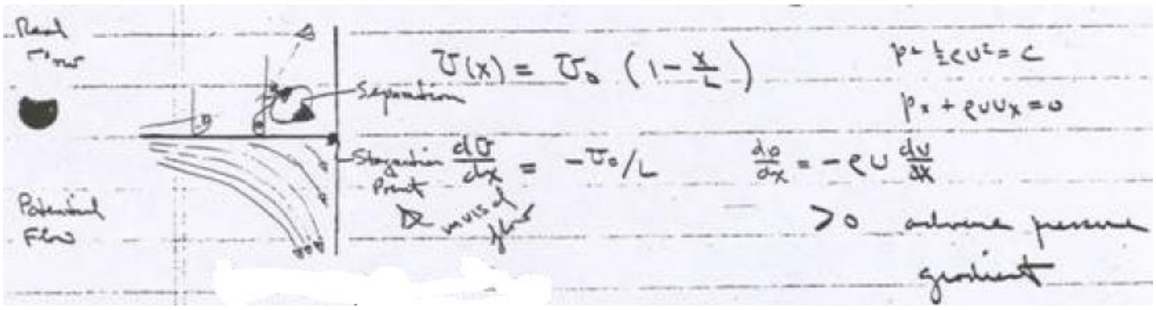


Fig. 10.3. The functions $F(\eta)$ and $G(\eta)$ for the velocity distribution in the boundary layer from eqns. (10.22) and (10.23)

Fig. 10.4. The one-parameter family of velocity profiles from eqn. (10.22)

Howarth linearly decelerating flow (example of exact solution of steady state 2D boundary layer)



Howarth proposed a linearly decelerating external velocity distribution $U(x) = U_0\left(1-\frac{x}{L}\right)$ as a theoretical model for laminar boundary layer study. Use Thwaites's method to compute:

a) X_{sep}
b) $C_f\left(\frac{x}{L}=0.1\right)$

Note $U_x = -U_0/L$

Solution

$$\theta^2 = \frac{0.45\nu}{U_0^6\left(1-\frac{x}{L}\right)^6}\int_0^x U_0^5\left(1-\frac{x}{L}\right)^5 dx = 0.075\frac{\nu L}{U_0}\left[\left(1-\frac{x}{L}\right)^{-6}-1\right]$$

can be evaluated for given L, Re_L

$$\lambda = \frac{\theta^2}{\nu}\frac{dU}{dx} = -0.075\left[\left(1-\frac{x}{L}\right)^{-6}-1\right]$$

$$\lambda_{sep} = -0.09 \Rightarrow \frac{X_{sep}}{L} = 0.123$$

3% higher than exact solution =0.1199

$C_f\left(\frac{x}{L} = 0.1\right)$ →, i.e., just before separation

$$\lambda = -0.0661$$

$$S(\lambda) = 0.099 = \frac{1}{2} C_f Re_\theta$$

$$C_f = \frac{2(0.099)}{Re_\theta}$$

Compute Re_θ in terms if Re_L

$$\theta^2 = 0.075 \frac{\nu L}{U_0}[(1-.1)^{-6} - 1] = 0.0661 \frac{\nu L}{U_0}$$

$$\frac{\theta^2}{L^2} = 0.0661 \frac{\nu}{U_{0L}} = \frac{0.0661}{Re_L}$$

$$\frac{\theta}{L} = \frac{0.257}{\sqrt{Re_L}}$$

$$Re_\theta = \frac{\theta}{L} Re_L = 0.257\sqrt{Re_L}$$

$$C_f = \frac{2(0.099)}{0.257\sqrt{Re_L}} = 0.77/\sqrt{Re_L}$$ → To complete solution must specify Re_L

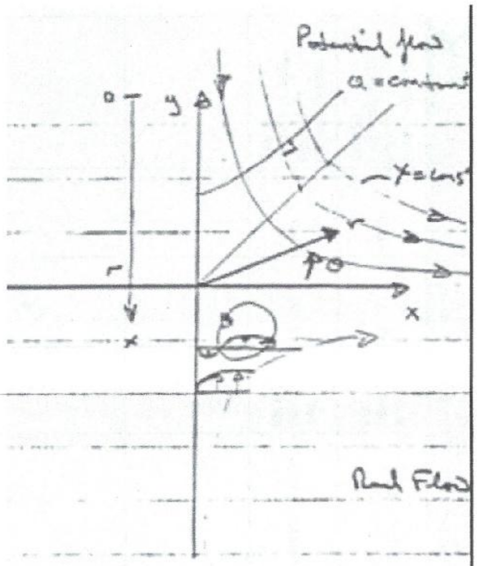
Consider the complex potential

$$F(z) = \frac{a}{2}z^2 = \frac{a}{2}r^2 e^{2i\theta}$$

$$\varphi = \text{Re}[F(z)] = \frac{a}{2}r^2 \cos 2\theta$$

$$\psi = \text{Im}[F(z)] = \frac{a}{2}r^2 \sin 2\theta$$

Orthogonal rectangular hyperbolas



φ: asymptotes y = ± x

ψ: asymptotes x=0, y=0

$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}$

$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$

$$\underline{V} = \nabla\varphi = \varphi_r \hat{e}_r + \frac{1}{r}\varphi_\theta \hat{e}_\theta$$

$$\left.\begin{aligned} v_r &= ar\cos 2\theta \\ v_\theta &= -ar\sin 2\theta \end{aligned}\right\} \quad \frac{\pi}{2} \le \theta \le 0 \text{ (flow direction as shown)}$$

$$\underline{V} = v_r(\cos\theta\hat{i} + \sin\theta\hat{j}) + v_\theta(-\sin\theta\hat{i} + \cos\theta\hat{j}) =$$
$$(v_r\cos\theta - v_\theta\sin\theta)\hat{i} + (v_r\sin\theta + v_\theta\cos\theta)\hat{j}$$

Potential flow slips along surface: (consider $\theta = 90^\circ$)

1) determine a such that $v_r = U_0$ at r=L, $\theta = 90^\circ$

$v_r = aL\cos(2\times 90) = U_0 \Rightarrow aL = -U_0$, i.e. $a = -\frac{U_0}{L}$

2) let $U(x) = v_r$ at x=L-r:

$\Rightarrow v_r = a(L-x)\cos(2\times 90) = U(x)$

$$Or:\ U(x) = -a(L-x) = \frac{U_0}{L}(L-x) = U_0(1-\frac{x}{L})$$

$\psi = Bxy \quad B = U/L$

$u = Bx = \psi_y$

$v = -By = -\psi_x$

$p + \frac{1}{2}\rho(u^2+v^2) = C$

$p + \frac{1}{2}\rho B^2(x^2+y^2) = C$

$p(0,0) = C = p_0$

$p = p_0 - \frac{1}{2}\rho\frac{U^2}{L^2}(x^2+y^2)$

$p_x = -\rho\frac{U^2}{L^2}x$

$p_y = -\rho\frac{U^2}{L^2}y$

$U_x = -\frac{U_0}{L}$

$p + \frac{1}{2}\rho U^2 = C$

$p_x + \rho UU_x = 0$

$p_x = -\rho UU_x = -\rho U_0(1-\frac{x}{L})(-\frac{U_0}{L}) = \rho\frac{U_0^2}{L}(1-\frac{x}{L})$

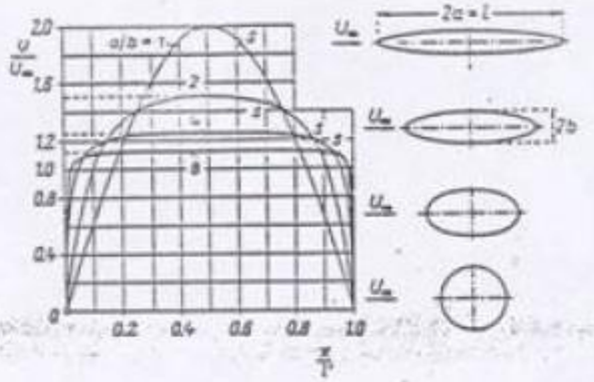


Fig. 10.9. Potential velocity distribution function on elliptical cylinders of slenderness $a/b = 1, 2, 4, 8$, the direction of the stream being parallel to the major axis

S = position of point of separation

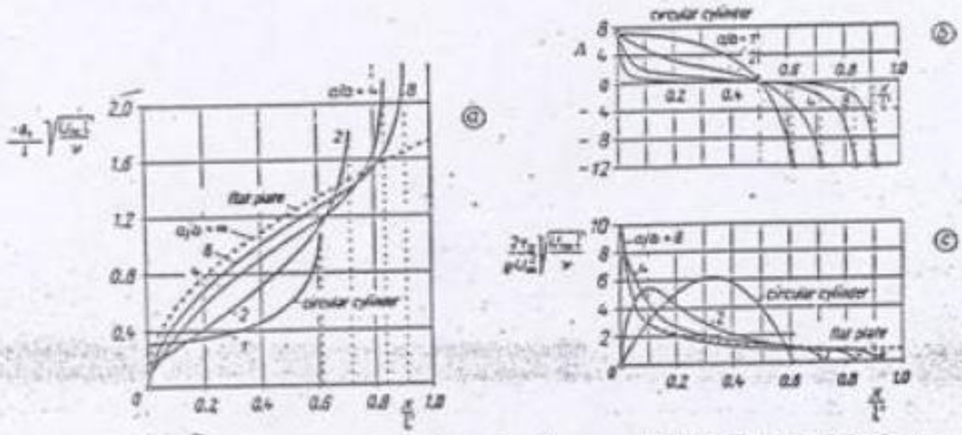


Fig. 10.10. Results of the calculation of boundary layers on elliptical cylinders of slenderness $a/b = 1, 2, 4, 8$, Fig. 10.9. a) displacement thickness of the boundary layer, b) shape factor c) shearing stress at the wall, $2l'$ = circumference of the ellipse; $a/b = 1$ circular cylinder; $a/b = \infty$ flat plate

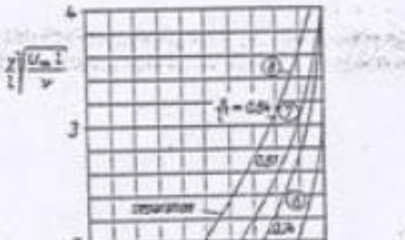


Fig. 10.11. Velocity profiles in the laminar boundary layer on an elliptical cylinder. Ratio of axes $a/b = 4$

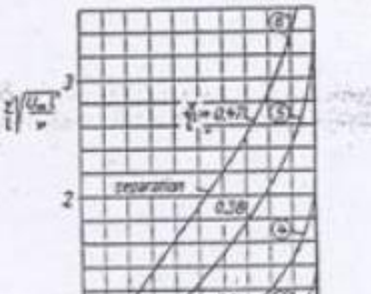


Fig. 10.12. Velocity profiles in the laminar boundary layer and potential velocity function for a Zhukovskii aerofoil J 015 of thickness ratio $d/l = 0.15$ at an angle of incidence $\alpha = 0$

Refer back to Table 4-6 (Lec 26) with regard to relative performance of integral methods

Alternative derivation momentum integral equation #1 (Currie)

$u_x + u_y = 0$

$(u^2)_x + (uv)_y = \mathrm{UU_x} + \nu u_{yy}$ $\qquad 2uu_x + u_y v + \underbrace{uu_y}_{-uu_x} = \text{ RHS}$

$$uu_x + vu_y = \text{ RHS}$$

$u(x,0) = 0 \qquad u(x,\delta) = U(x) \quad \mu u_y(x,0) = \tau_0 = \tau_w$

$u_y(x,\delta) = 0 \qquad \tau = \mu u_y \Rightarrow \nu u_{yy} = \tau_y/\rho$

$$\int_0^\delta [(u^2)_x + (uv)_y] dy = \int_0^\delta [UU_x + \nu\tau_y] dy$$

$$\int_0^\delta (u^2)_x \; dy + U \underbrace{v(x,\delta)}_{-\int_0^\delta u_x\, dy} = U_x \int_0^\delta U \; dy - \frac{\tau_0}{\rho}$$

$U \neq f(y)$ but retained in integral to obtain desired form integral equation

$$\int_0^\delta (u^2)_x \; dy - U \int_0^\delta u_x \; dy = U_x \int_0^\delta U \; dy - \frac{\tau_0}{\rho}$$

Leibniz rule: $\int_{a(x)}^{b(x)} u_x \; dy = \frac{d}{dx}\int_a^b u \; dy + u(x,a)\frac{da}{dx} - u(x,b)\frac{db}{dx}$

$$\int_0^\delta (u^2)_x \; dy = \frac{d}{dx}\int_0^\delta u^2 \; dy - U^2\delta_x$$

$$\int_0^\delta u_x \; dy = \frac{d}{dx}\int_0^\delta u \; dy - U\delta_x$$

$$\frac{d}{dx}\int_0^{\delta} u^2\ dy - \cancel{U^2\delta_x} - \underbrace{U\frac{d}{dx}\int_0^{\delta} u\ dy}_{=\frac{d}{dx}\int_0^{\delta} Uu\,dy - U_x\int_0^{\delta} u\,dy} + \cancel{U^2\delta_x}$$

$$= U_x\int_0^{\delta} U\ dy - \frac{\tau_0}{\rho}$$

$$\frac{d}{dx}\int_0^{\delta} u^2\ dy - \frac{d}{dx}\int_0^{\delta} Uu\ dy + U_x\int_0^{\delta} u\ dy = U_x\int_0^{\delta} U\ dy - \frac{\tau_0}{\rho}$$

$$\frac{d}{dx}\int_0^{\delta} u(U-u)\ dy + U_x\int_0^{\delta}(U-u)\ dy = \frac{\tau_0}{\rho}$$

$$\frac{d}{dx}\left[U^2\underbrace{\int_0^{\infty}\left(\frac{u}{U}\left(1-\frac{u}{U}\right)\right)dy}_{\theta}\right] + U_xU\underbrace{\int_0^{\infty}\left(1-\frac{u}{U}\right)dy}_{\delta^*} = \frac{\tau_0}{\rho}$$

Since for $\delta \to \infty$, $U - u = 0$ upper limit can be taken as ∞.

$$\frac{d}{dx}(U^2\theta) + U_xU\delta^* = \frac{\tau_0}{\rho}$$

or $2UU_x\theta + U^2\theta_x + U_xU\delta^* = \frac{\tau_0}{\rho}$

$$\theta_x + (2\theta + \delta^*)\frac{U_x}{U} = \frac{\tau_0}{\rho U^2} = \frac{C_f}{2}$$

$$\theta_x + (2 + H)\frac{\theta}{U}U_x = \frac{C_f}{2} \qquad H = \frac{\delta^*}{\theta}$$

Assume $u(x,y) \Rightarrow \theta,\ \delta^*,\ \tau_0 \Rightarrow f(\delta)$ then solve momentum integral for $\delta(x)$

Karman–Pohlhausen 4th order velocity profile in classical approach, which also uses empirical correlations.

Karman–Pohlhausen Method

$$\frac{u}{U} = a + b\eta + c\eta^2 + d\eta^3 + e\eta^4 \qquad \eta(x,y) = \frac{y}{\delta(x)}$$

where $a - e = f(x)$, i.e., self-similarity not possible

$$u(x,0) = 0 \qquad u(x,\delta) = U(x)$$

$$u_y(x,\delta) = 0 \qquad \underbrace{u_{yy}(x,0) = -\frac{U(x)}{\nu} U_x}_{\text{BL momentum}|_{y=0}} \qquad u_{yy}(x,\delta) = 0$$

or in terms of $u/U = f(\eta)$

1 2

$$\frac{u}{U} = 0 \quad \left(\frac{u}{U}\right)_{\eta\eta} = -\frac{\delta^2}{\nu} U_x = -\Lambda(x) \quad \eta = 0$$

3 4 5

$$\frac{u}{U} = 1 \quad \left(\frac{u}{U}\right)_{\eta} = \left(\frac{u}{U}\right)_{\eta\eta} = 0 \qquad \eta = 1$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y} \qquad \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 = \frac{\partial}{\partial \eta^2}\delta^{-2}$$

$$u_{yy} = \frac{\partial}{\partial \eta^2}\delta^{-2}(uU) = u_{\eta\eta}\,\delta^{-2}U = -\frac{U(x)}{\nu}U_x$$

$$u_{\eta\eta} = -\frac{\cancel{U}U_x}{\nu}\frac{\delta^2}{\cancel{U}} = -\frac{\delta^2 U_x}{\nu} = -\Lambda(x)$$

$$\underbrace{-\Lambda(x)}_{\text{diensionless measure } p_x \text{ outer flow}}$$

$0 \;=\; a$ 1

$-\Lambda = 2c$ 2 $\left(\frac{u}{U}\right)_\eta = b + 2c\eta + 3d\eta^2 + 4e\eta^3$

$1 = a + b + c + d + e$ 3 $\left(\frac{u}{U}\right)_{\eta\eta} = 2c + 6d\eta + 12e\eta^2|_{\eta=0} = 2c$

$0 = b + 2c + 3d + 4e$ 4

$0 = 2c + 6d + 12e$ 5

$a = 0 \qquad b = 2 + \frac{\Lambda}{6} \qquad c = -\frac{\Lambda}{2} \qquad d = -2 + \frac{\Lambda}{2} \qquad e = 1 - \frac{\Lambda}{6}$

$0 = -\Lambda + 6d + 12e$

$$\mathrm{d} = \frac{\Lambda - 12e}{6} = \frac{\Lambda}{6} - 2e = \frac{\Lambda}{6} - 2 + \frac{2\Lambda}{6} = -2 \; + \Lambda/2$$

$$0 = b - \Lambda + \frac{\Lambda}{2} - 6e + 4e$$

$$b = \frac{\Lambda}{2} + 2e = \frac{\Lambda}{2} + 2 - \frac{\Lambda}{3} = 2 + \frac{\Lambda}{6}$$

$$1 = \cancel{\frac{\Lambda}{2}} + \cancel{2e} - \cancel{\frac{\Lambda}{2}} + \frac{\Lambda}{6} - \cancel{2e} + e$$

$$e = 1 - \frac{\Lambda}{6}$$

$$\frac{u}{U} = \left(2 + \frac{\Lambda}{6}\right)\eta - \frac{\Lambda}{2}\eta^2 + \left(-2 + \frac{\Lambda}{2}\right)\eta^3 + \left(1 - \frac{\Lambda}{6}\right)\eta^4$$

$$= (2\eta - 2\eta^3 + \eta^4) + \frac{\Lambda}{6}(\eta - 3\eta^2 + 3\eta^3 - \eta^4)$$

$$= 1 - (1+\eta)(1-\eta)^3 + \Lambda\frac{1}{6}\eta(1-\eta)^3$$

$$\frac{u}{U} = F(\eta) + \Lambda G(\eta)$$

$(1-\eta)(1-\eta)^2 = (1-2\eta+\eta^2)(1-\eta) = 1-2\eta+\eta^2-\eta+2\eta^2-\eta^3 \;= 1-3\eta+3\eta^2-\eta^3$

$(1-3\eta+3\eta^2-\eta^3)(1+\eta) = 1-3\eta+\cancel{3\eta^2}-\eta^3+\eta-\cancel{3\eta^2}+3\eta^3-\eta^4 \;= 1-2\eta+2\eta^3-\eta^4$

$$1-RHS = 2\eta-2\eta^3+\eta^4$$

$F(\eta)$ monotonically increasing

$f(\eta) \quad 0 \le F \le 1$

$G(\eta)$ $0|_{\eta=0}$ to max $= 0.0166|_{\eta=0.25}$

to $0|_{\eta=1}$

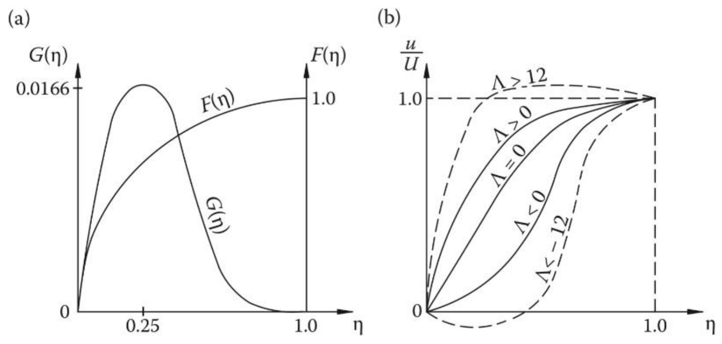


FIGURE 9.6
(a) Form of functions *F*(η) and *G*(η), and (b) velocity profiles for various values of the parameter Λ(*x*).

Λ=0 u/U= F(η) 4th order polynomial approximation, Blasius

> 12 overshoot not physical $\therefore$ Λ restricted < 12

< -12 reverse flow, i.e., separation, which violates BL theory

Conclusion: $-12 < \Lambda(x) < 12$

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

$\eta = \frac{y}{\delta}, \quad d\eta = \frac{dy}{\delta} \quad y = 0 \ \Rightarrow \eta = 0, \quad y = \delta \quad \eta = 1$

$$= \delta \int_0^1 \left(1 - \frac{u}{U}\right) d\,\eta$$

$$= \delta \int_0^1 \left[(1+\eta)(1-\eta)^3 - \frac{\Lambda}{6}\eta(1-\eta)^3\right] d\eta = \delta\left(\frac{3}{10} - \frac{\Lambda}{120}\right)$$

$$\theta = \delta \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\,\eta = \delta\left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072}\right)$$

$$\tau_0 = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu \frac{U}{\delta}\frac{d}{d\eta}\left.\left(\frac{u}{U}\right)\right|_{\eta=0} =$$

$$\mu \frac{U}{\delta}\frac{\partial}{\partial \eta}\left.\left[1 - (1+\eta)(1-\eta)^3 + \frac{\Lambda}{6}\eta(1-\eta)^3\right]\right|_{\eta=0} = \mu\frac{U}{\delta}\left(2 + \frac{\Lambda}{6}\right)$$

$\delta^*,\ \theta,\ \tau_0 = f(\delta)$ $\quad \delta$ determined $\underbrace{\text{momentum integral equation}}$ ↓

$$\frac{U\delta}{\nu}\,\theta_x + (2\theta + \delta^*)\frac{\theta}{\nu}U_x = \frac{\tau_0}{\mu}\frac{\theta}{U} \qquad \frac{U\theta}{\nu}\times$$

$$\frac{1}{2}U\frac{d}{dx}\left(\frac{\theta^2}{\nu}\right) + \left(2 + \frac{\delta^*}{\theta}\right)\frac{\theta^2}{\nu}U_x = \frac{\tau_0\theta}{\mu U}$$

$$\Lambda = \frac{\delta^2}{\nu}U_x, \text{ i.e., } \frac{\theta^2}{\nu}U_x = \frac{\theta^2}{\delta^2}\Lambda = \left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072}\right)^2 \Lambda = K(x)$$

Also $\dfrac{\delta^*}{\theta} = \dfrac{\left(\frac{3}{10} - \frac{\Lambda}{120}\right)}{\left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072}\right)} = f(K)$

$f = f\big(\Lambda(x)\big), \quad K = K(x) \;\therefore\; f = f(K)$

$$\frac{\tau_0 \theta}{\mu U} = \underbrace{\left(2 + \frac{\Lambda}{6}\right)}_{\frac{\tau_0 \delta}{\mu U}} \underbrace{\left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072}\right)}_{\theta/\delta} = g(K)$$

Functionally $g(K)$ same reasoning f(K)

$$\frac{1}{2} U \frac{d}{dx}\left(\frac{\theta^2}{\nu}\right) + [2 + f(K)]K = g(K) \qquad K = \frac{\theta^2}{\nu} U_x$$

Let $Z = \dfrac{\theta^2}{\nu}$ = new dependent variable i.e. $K = Z U_x$

$$U Z_x = 2\big[g(K) - \big(2 + f(K)\big)K\big] = H(K)$$

$$H(K) = 2\left\{\left(2 + \frac{\Lambda}{6}\right)\left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9,072}\right) - \left[2 + \frac{\left(\frac{3}{10} - \frac{\Lambda}{120}\right)}{\left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9,072}\right)}\right]\left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9,072}\right)^2 \Lambda\right\}$$

$$= 2\left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9,072}\right)\left[2 - \frac{116}{315}\Lambda + \left(\frac{2}{945} + \frac{1}{120}\right)\Lambda^2 + \frac{2}{9,072}\Lambda^3\right]$$

$$K = \left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9,072}\right)^2 \Lambda$$

K *and* $H(K) = f\big(\Lambda(x)\big)$

$H(K) \approx 0.47 - 6K$

$U Z_x = 0.47 - 6K$

$= 0.47 - 6 Z U_x$

$$U \frac{dZ}{dx} + 6Z \frac{dU}{dx} = 0.47$$

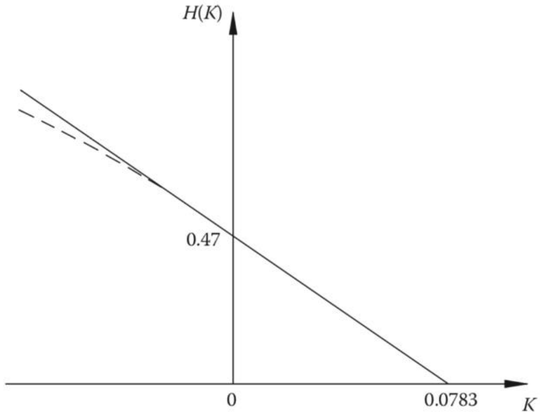


FIGURE 9.7
Exact form of the function $H(K)$ (solid line) and straight-line approximation (dashed line).

$$\frac{1}{U^5}\frac{d}{dx}(ZU^6) = \frac{1}{U^5}\left[U^6\frac{dZ}{dx} + 6U^5ZU_x\right]$$

$$= UZ_x + 6ZU_x = 0.47$$

$$d(ZU^6) = 0.47\,U^5\,dx$$

$$ZU^6 = 0.47\int_0^x U^5\ dx$$

$Z = \frac{\theta^2}{\nu} = \frac{0.47}{U^6}\int_0^x U^5\ dx$ i.e. $\theta^2(x) = \frac{0.47\,\nu}{U(x)^6}\int_0^x U(x)^5\ dx$

where $U(x)$ potential flow solution geometry of interest

Solution procedure:

1. $U(x)$
2. $\theta(x)$
3. $\Lambda(x)$ from $\underbrace{\left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072}\right)^2}_{\frac{\theta^2}{\delta^2}}\Lambda = \frac{\theta^2}{\nu}U_x$ note $\Lambda(x) = \frac{\delta^2}{\nu}U_x$
4. $\delta(x) = \frac{\theta(x)}{\left(\frac{37}{315}-\frac{\Lambda}{945}-\frac{\Lambda^2}{9072}\right)}$
5. $\delta^*(x) = \delta\left(\frac{3}{10} - \frac{\Lambda}{120}\right)$
6. $\frac{u}{U} = f(\eta), \quad \eta = \frac{y}{\delta(x)}$
7. $\tau_0 = \mu\frac{U}{\delta(x)}\left(2 + \frac{\Lambda}{6}\right)$

Inverse problem: for specified $\Lambda(x) \Rightarrow U(x) \Rightarrow$ type of geometry

Example: flat surface, i.e.,

(1) U = constant, $U_x = 0$

(2) $\theta^2 = 0.47 \frac{\nu x}{U}$ $\quad \frac{\theta}{x} = \frac{0.686}{\sqrt{Re}}$ $\quad Re = \frac{Ux}{\nu}$

(3) $\Lambda = 0$

(4) $\delta = \frac{315}{37}\theta \Rightarrow \frac{\delta}{x} = \frac{5.84}{\sqrt{Re}}$ $\quad \delta^{-1} = \frac{\sqrt{Ux/\nu}}{5.84x}$

(5) $\delta^* = \frac{3}{10}\delta \Rightarrow \frac{\delta^*}{x} = \frac{1.75}{\sqrt{Re}}$

(6) $\tau_0 = \mu \frac{2U}{\delta(x)} \Rightarrow \frac{\tau_0}{\frac{1}{2}\rho U^2} = \frac{0.686}{\sqrt{Re}}$

$$\frac{\tau_0}{\frac{1}{2}\rho U^2} = \frac{2\mu U}{\frac{1}{2}\rho U^2}\delta^{-1} = \frac{4\nu}{U}\frac{U^{1/2}x^{-1/2}}{5.84\,\nu^{1/2}} = \frac{0.685\,\nu^{1/2}}{U^{1/2}x^{1/2}} = \frac{0.685}{\sqrt{Re}}$$

τ_0 within 3% Blasius

i.e. 0.685 vs. .686

whereas using 2nd order polynomial value was 0.73

Alternative derivation momentum integral equation #2 (Panton)

h = outside BL where $u = U_e$ and u_y, u_{yy}, etc. = 0

$$\int_0^h [uu_x + vu_y]dy = \int_0^h \left[U_e U_{e_x} + \frac{\mu}{\rho}u_{yy}\right] dy$$

$$\int_0^h [uu_x - U_e U_{e_x} + vu_y]dy = \left[\frac{\mu}{\rho}u_y\right]_0^h = -\frac{\tau_0}{\rho}$$

$$v = \int_0^y v_y\, dy = -\int_0^y u_x\, dy$$

integration by parts

$$\int_0^h w\, dz = [wz]_0^h - \int_0^h z\, dw$$

$$w = -\int_0^{y=u} u_x\, dy \quad dw = -u_x\, dy$$

$$dz = u_y\, dy \quad z = u$$

$$\int_0^h v\, u_y\, dy = \int_0^h \left(-\int_0^y u_x\, dy\right) u_y\, dy$$

$$= \underbrace{-U_e \int_0^h u_x\, dy}_{[wz]_0^h} + \underbrace{\int_0^h u\, u_x\, dy}_{-\int_0^h z\, dw}$$

$$\int_0^h [uu_x - U_e U_{e_x} - U_e u_x + uu_x]dy = -\frac{\tau_0}{\rho} \qquad + \text{ and } -uU_{e_x}$$

$$-\int_0^h [u(U_e - u)_x + (U_e - u)u_x]dy - \int_0^h (U_e - u)U_{e_x}\, dy = -\frac{\tau_0}{\rho}$$

$$\text{or } \frac{d}{dx}\left[U_e^2 \underbrace{\int_0^h \left(\frac{u}{U_e}\left(1 - \frac{u}{U_e}\right)\right) dy}_{\theta}\right] + U_e U_{e_x} \underbrace{\int_0^h \left(1 - \frac{u}{U_e}\right) dy}_{\delta^*} = \frac{\tau_0}{\rho}$$

$$\frac{d}{dx}[u(U_e - u)] = u_x(U_e - u) + u\left(U_{e_x} - u_x\right)$$

$$\frac{d}{dx}(U_e^2\theta) + U_e U_{e_x}\delta^* = \frac{\tau_0}{\rho}$$

Karman–Pohlhausen

$$u^* = \frac{u}{u_e} = a + b\eta + c\eta^2 + d\eta^3 \qquad \eta = \frac{y}{\delta(x)}$$

$$u = 0, \quad -U_e U_{e_x} = \nu u_{yy}, \quad u_{yyy} = 0 \quad y = 0$$

$$u = U_e, \quad u_y = 0, \quad u_{yy} = 0 \quad y = \delta$$

$\frac{\partial^n u}{\partial y^n} = 0 \qquad n > 2$ for u/U_e polynomial $>$ 3rd order

Example: Blasius BL $U_e = U_0$

$$u^* = \frac{3}{2}\eta - \frac{1}{2}\eta^3$$

$$\delta^* = \delta \int_0^1 (1 - u^*)\ d\eta = \frac{3}{8}\delta \qquad \theta = \delta \int_0^1 u^*(1 - u^*)\ d\eta = \frac{117}{840}\delta$$

$$\frac{\tau_0}{\rho} = \nu u_y|_{y=0} = \nu \frac{3}{2}\frac{U_0}{\delta}$$

$$U_e^2\, \theta_x = \frac{\tau_0}{\rho}$$

$$U_e^2 \frac{117}{840}\delta_x = \frac{3}{2}\frac{\nu U_0}{\delta} \Rightarrow \delta = \sqrt{\frac{840}{39}}\sqrt{\frac{\nu x}{U_0}} = 4.64\sqrt{\frac{\nu x}{U_0}}$$

$\frac{U_e^2\, 117}{840}\, \delta\, d\delta = \frac{3}{2}\nu U_0\, dx$ vs. 4.9 exact value

$$\frac{U_e^2\, 117}{840}\frac{1}{2}\delta^2 = \frac{3}{2}\nu U_0\, x$$

$$\delta^2 = \frac{3 \times 840}{117}\frac{\nu x}{U_e} \qquad \delta = 4.64\sqrt{\frac{\nu x}{U_e}}$$

Alternative derivation momentum integral equation #3 (Kundu)

$$uu_x + vu_y = u_e u_{e,x} + \frac{1}{\rho}\tau_y$$

$$u(u_x + v_y) + uu_x + vu_y = (u^2)_x + (vu)_y = u_e u_{e_x} + \frac{1}{\rho}\tau_y$$

$$2\mathrm{uu_x} + \mathrm{v_y u} + \mathrm{vu_y} = \mathrm{u_x u} + \mathrm{vu_y} + \mathrm{u}(\mathrm{u_x} + \mathrm{v_y})$$

$$\int_0^\infty [(u^2)_x + (uv)_y - u_e u_{e_x}]dy = \frac{1}{\rho}\int_0^\infty \tau_y \, dy$$

$$\int_0^\infty [(u^2)_x - u_e u_{e_x}]dy + u_e v_\infty = -\frac{\tau_w}{\rho}$$

$$\int_0^\infty (u_x + v_y) \, dy = 0 \qquad \int_0^\infty u_x \, dy = -\int_0^\infty v_y \, dy = -v|_0^\infty = -v_\infty$$

$$\int_0^\infty [(u^2)_x - u_e u_{e_x} - u_e u_x]dy = -\frac{\tau_w}{\rho}$$

$$u_e \int_0^\infty u_x \, dy = u_e \frac{d}{dx}\int_0^\infty u \, dy = \frac{d}{dx}\left[u_e \int_0^\infty u \, dy\right] - u_{e_x}\int_0^\infty u \, dy$$

$$\frac{d}{dx}\int_0^\infty (u^2 - u_e u)dy + u_{e_x}\int_0^\infty (u - u_e) \, dy = -\frac{\tau_w}{\rho}$$

$$\frac{\tau_w}{\rho} = \frac{d}{dx} u_e^2 \underbrace{\int_0^\infty \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy}_{\theta} + u_e u_{e_x} \underbrace{\int_0^\infty \left(1 - \frac{u}{u_e}\right) dy}_{\delta^*}$$

$\frac{\tau_w}{\rho} = \frac{d}{dx}(u_e^2\theta) + u_e u_{e_x}\delta^*$ note: u_e and u_{e_x} $f(x)$ only

single ODE relating unknowns θ, δ^* and τ_w

Example: momentum integral, $U_e(x) = \frac{U_0}{L} x$

Accelerating flow i.e. stagnation point flow $\underbrace{FS\ with\ m\ = 1}_{U=kx}$

$$g(x) = \left[\frac{2}{m+1}\frac{\nu x}{U}\right]^{1/2} = \left[\frac{\nu}{k}\right]^{1/2} = \left[\frac{\nu L}{U_0}\right]^{1/2}$$

$$\delta(x) = \sqrt{\frac{\nu x}{U_e}} = \left(\frac{\nu L}{U_0}\right)^{1/2} = \text{constant} \qquad \frac{u}{u_e} = f'(\eta)$$

$$\eta = \frac{y}{\delta}$$

$\theta = \int_0^\infty \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy = \delta \int_0^\infty f'(1-f')\ d\eta\ =\ \delta \times$ constant and

$\delta^* = \delta \times$ constant, i.e., both θ and δ^* integrals velocity profile and $\therefore \propto \delta$

$$\psi = [\nu x U_e(x)]^{1/2} f(\eta)$$

$$u = \psi_y = \frac{\left[\nu x \frac{U_0}{L} x\right]^{1/2}}{[\nu L/U_0]^{1/2}} f'(\eta) = \frac{U_0 x}{L} f'(\eta)\ = U_e\, f'(\eta)$$

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left[\frac{U_0^2 x^2}{L^2}\theta\right] + \frac{U_0 x}{L}\frac{d}{dx}\left[\frac{U_0 x}{L}\right]\delta^* = \frac{2U_0^2 x}{L^2}\theta + \frac{U_0^2 x}{L^2}\delta^*$$

$\frac{\tau_w}{\frac{1}{2}\rho U_0^2} = \left(\frac{4\theta + 2\delta^*}{L}\right)\frac{x}{L} = ax$ i.e. increases linearly

One equation, three unknowns: $\tau_w,\ \theta,\ \delta^*$

Can use Thwaites method to solve (or Karman–Pohlhausen)

EXAMPLE 10.6

Use Thwaites' method to estimate the momentum thickness, displacement thickness, and wall shear stress of the Blasius boundary layer with $\theta_0 = 0$ at $x = 0$.

Solution

The solution plan is to use (10.50) to obtain θ. Then, because $dU_e/dx = 0$ for the Blasius boundary layer, $\lambda = 0$ at all downstream locations and the remaining boundary-layer parameters can be determined from the θ results, (10.45), (10.46), and Table 10.2. The first step is setting $U_e = U =$ constant in (10.50) with $\theta_0 = 0$:

$$\theta^2 = \frac{0.45\nu}{U^6}\int_0^x U^5 dx = \frac{0.45\nu}{U}x, \quad \text{or} \quad \theta = 0.671\sqrt{\frac{\nu x}{U}}.$$

This approximate answer is 1% higher than the Blasius-solution value. For $\lambda = 0$, the tabulated shape factor is $H(0) = 2.61$, so:

$$\delta^* = \theta\left(\frac{\delta^*}{\theta}\right) = \theta H(0) = 0.671\sqrt{\frac{\nu x}{U}}(2.61) = 1.75\sqrt{\frac{\nu x}{U}}.$$

This approximate answer is also 1% higher than the Blasius-solution value. For $\lambda = 0$, the shear correlation value is $l(0) = 0.220$, so:

$$\tau_w = \mu\frac{U}{\theta}l(0) = \frac{\mu U}{0.671\sqrt{\nu x/U}}(0.220) = \frac{1}{2}\rho U^2(0.656)\sqrt{\frac{\nu}{Ux}},$$

which implies a skin friction coefficient of:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.656}{\sqrt{\text{Re}_x}},$$

which is 1.2% below the Blasius-solution value.

Note: Same Karman-Pohlhausen since we very similar correlations

$H(K) = .47 - 6K$ or $F(\lambda) = .45 - 6\lambda$

$K = \frac{\theta^2}{\nu} U_x$, $\lambda = K$

Example: Kundu

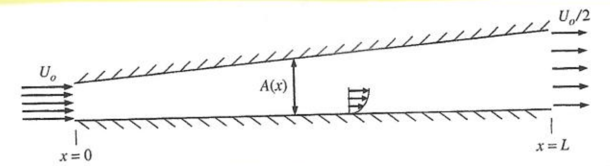


FIGURE 10.10 A simple two-dimensional diffuser of length L intended to slow the incoming flow to half its speed by doubling the flow area. The resulting adverse pressure gradient in the diffuser influences the character of the boundary layers that develop on the diffuser's inner surfaces, especially when these boundary layers are laminar.

Assume outer flow uniform: $U_1 A_1 = U_e(x) A(x)$

$A(x) = A_1 (1 + x/L)$

$$\theta^2 = \frac{.45\nu}{U_e^6}\int_0^x U_e^5 \, dx = \frac{.45\nu}{U_1}(1+x/L)^6 \int_0^x \left(1+\frac{x}{L}\right)^{-5} dx \qquad U_e(x) = U_1 (1 + x/L)^{-1}$$

$$= \frac{.45\nu}{U_1}\left(1+\frac{x}{L}\right)^6 \left[1 - \left(1+\frac{x}{L}\right)^{-4}\right]\frac{L}{4}$$

$$\lambda = \frac{\theta^2}{\nu} U_{e_x} = \theta^2 \times \frac{U_{e_x}}{\nu} \qquad \frac{U_{e_x}}{\nu} = -\left(\frac{U_1}{\nu L}\right)\left(1+\frac{x}{L}\right)^{-2}$$

$$= -\frac{.45}{4}\left[\left(1+\frac{x}{L}\right)^4 - 1\right]$$

x/L	λ
0	0
0.05	−0.02424
0.10	−0.05221
0.15	−0.08426
0.20	−0.12078

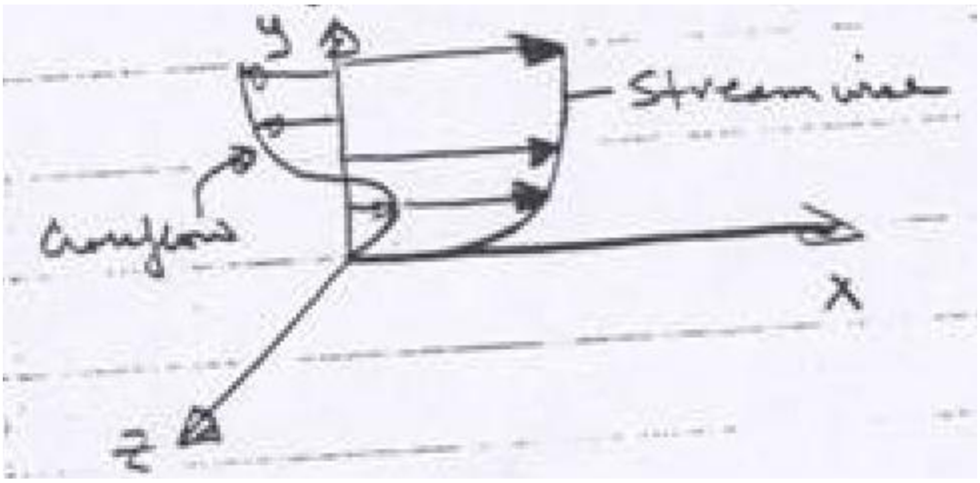
→ $\frac{x}{L} = .16 \quad << L$

$\lambda = -.09$

i.e. separation

3-D Integral methods

Momentum integral methods perform well (i.e. compare well with experimental data) for a large class of both laminar and turbulent 2D flows. However, for ***3D flows they do not***, primarily due to the inability of correlating the crossflow velocity components.



The cross flow is driven by $\dfrac{\partial p}{\partial z}$, which is imposed on BL from the outer potential flow U(x,z).

3-D boundary layer equations

$$u_x + v_y + w_z = 0;$$
$$uu_x + vu_y + wu_z = -\frac{\partial}{\partial x}(p/\rho) + \nu u_{yy} - \frac{\partial}{\partial y}(\overline{u'v'})$$
$$uw_x + vw_y + ww_z = -\frac{\partial}{\partial z}(p/\rho) + \nu w_{yy} - \frac{\partial}{\partial y}(\overline{v'w'})$$
$$+ \quad \text{closure equations}$$

Differential methods have been developed for this reason as well as for extensions to more complex and non-thin boundary layer flows.