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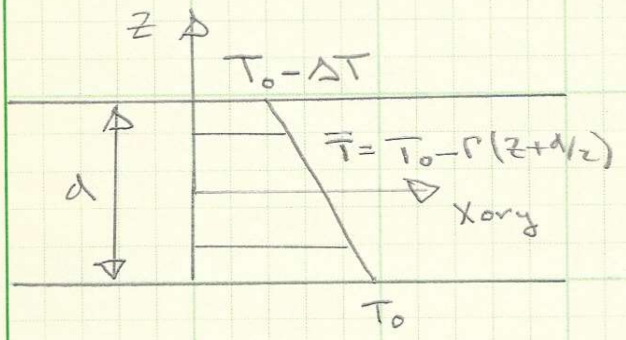
Thermal Instability: Benard Problem

Consider a layer of fluid between two isothermal walls with lower wall at higher temperature than upper wall.



BC:

$\tilde{u}(\pm d/2) = 0$

$\tilde{T}(d/2) = T_0 - \Delta T$

$\tilde{T}(-d/2) = T_0$

total flow variables = basic state + perturbation

Boussinesq Approximation

Assume $M < .3$, $\rho \neq \rho(z)$ due to large hydrostatic pressure, & small ΔT such that constant properties $\rho, \mu, k,$ & C_p, except

$\rho = \rho_0 [1 - \alpha (T - T_0)]$ in gravity term in vertical momentum equation.

α = thermal expansion coefficient $= -\frac{1}{\rho} \frac{\partial \rho}{\partial T}$

$\alpha_{gas} = 3 \times 10^{-3}\ K^{-1}$ $\quad \alpha_{liquid} = 5 \times 10^{-4}\ K^{-1}$

$\Delta T = 10°C \Rightarrow \Delta\rho \sim 3\%\ \rho_0$

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$$\nabla \cdot \tilde{\underline{u}} = 0$$

$$\frac{D\tilde{u}_i}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \frac{\tilde{\rho} g}{\rho_0}\delta_{i3} + \nu \nabla^2 \tilde{\underline{u}}$$

$$\frac{D\tilde{T}}{Dt} = \kappa \nabla^2 \tilde{T}$$

$\kappa = k/\rho_0 C_p$ = thermal diffusivity

$\nu = \mu/\rho_0$

$\tilde{u}_i = 0 + u_i(\underline{x}, t)$

$\tilde{T} = \bar{T}(z) + T(\underline{x}, t)$

$\tilde{p} = \bar{p}(z) + p(\underline{x}, t)$

$\tilde{\rho} = \bar{\rho}(z) + \rho(\underline{x}, t)$ but only in w equation

Over bar = basic/initial State

squiggle = perturbation

Basic/Initial State: no motion

$\bar{u}(z) = 0$!

$u_i = T = p = 0$

$$0 = \frac{-1}{\rho_0}\frac{\partial \bar{p}}{\partial x_i} - g[1 - \alpha(\bar{T} - T_0)]\delta_{i3}$$ hydro static pressure variation

$$0 = \kappa \nabla^2 \bar{T}$$ thermal conduction

$$\bar{T} = T_0 - \Gamma(z + d/2) \qquad \Gamma = \Delta T/d = \frac{d\bar{T}}{dz}$$ linear temperature variation

$\bar{\rho}/\rho_0 = 1 + \alpha \Gamma d$

Perturbation Equations

Substitute $\tilde{u}_i$, $\tilde{T}$, $\tilde{p}$ into GOE:

$$\frac{Du_i}{Dt} = -\frac{1}{\rho_0}\frac{\partial}{\partial x_i}(\bar{p} + p) - g[1 - \alpha(\bar{T} - T - T_0)]\delta_{i3} + \nu \nabla^2 u_i$$

$$\frac{\partial T}{\partial t} + u_i \frac{\partial}{\partial x_i}(\bar{T} - T) = \kappa \nabla^2(\bar{T} + T)$$

Substract basic state equations &
neglect products of perturbations:

$$\frac{\partial u_i}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + g\alpha T\delta_{i3} + \nu\nabla^2 u_i$$

Convection terms = 0 in momentum equations

$$\frac{\partial T}{\partial t} - w\Gamma = \kappa\nabla^2 T$$

$$u_i\frac{\partial \bar{T}}{\partial x_i} = w\frac{d\bar{T}}{dz} = -w\Gamma \Rightarrow w = \frac{\kappa T/d^2}{\Gamma} = \frac{\kappa\Gamma/d}{\Gamma} = \frac{\kappa}{d}$$

Rewrite in terms of w & T only by

① taking Laplacian of w $(i=3)$ momentum equation

$$\frac{\partial}{\partial t}(\nabla^2 w) = -\frac{1}{\rho_0}\nabla^2\frac{\partial p}{\partial z} + g\alpha\nabla^2 T + \nu\nabla^4 w$$

② take divergence $\frac{\partial u_i}{\partial t}$ & use $\frac{\partial u_i}{\partial x_i} = 0$

$$0 = \underbrace{-\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x_i^2} + g\alpha\frac{\partial T}{\partial x_i}\delta_{i3}}$$

$$\frac{\partial}{\partial z}(\nabla) = 0 = -\frac{1}{\rho_0}\nabla^2\frac{\partial p}{\partial z} + g\alpha\frac{\partial^2 T}{\partial z^2}$$

③ Substitute ② into ①

$$\frac{\partial}{\partial t}(\nabla^2 w) = g\alpha\nabla_H^2 T + \nu\nabla^4 w$$

$$\nabla_H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

BC: $u_i(\pm d/2) = 0$, $T(\pm d/2) = 0$

Continuity : $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\Big|_{z=\pm d/2} = 0$

ie $\frac{\partial w}{\partial z}\Big|_{z=\pm d/2} = 0$

$w = \frac{\partial w}{\partial z} = T = 0$ on $z = \pm d/2$

Nondimensionalize equations

$t' = \frac{K}{d^2} t$ $\quad \underline{x}' = \underline{x}/d$ $\quad$ drop '

$\left(\frac{\partial}{\partial t} - \nabla^2\right) T = \frac{\Gamma d^2}{K} w$ $\qquad Pr = \nu/K$

$\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right) \nabla^2 w = \frac{g\alpha d^2}{\nu} \nabla^2_H T$

$w = \frac{\partial w}{\partial z} = T = 0$ $\quad z = \pm 1/2$

<u>Method of Normal Modes</u>

$w = \hat{w}(z)\, e^{ikx + ily + \sigma t}$

$T = \hat{T}(z)\, e^{ikx + ily + \sigma t}$

k, l = real = wave number of sinusoidal perturbation superimposed on basic/initial state

$\underline{K} = k\hat{i} + l\hat{j}$

$i\,\underline{K}\cdot\underline{x} = ikx + ily$

$\sigma = \sigma_r + i\sigma_i$ $\quad \sigma_i \neq 0 \Rightarrow$ travelling waves

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$\sigma_r < 0$ stable = disturbance decays

$\sigma_r = 0$ neutral = marginal state

$\sigma_r > 0$ unstable = disturbance grows

$\frac{\partial}{\partial t} = \sigma \quad \nabla_H^2 = K = \sqrt{l^2 + \alpha^2} \quad \nabla^2 = \frac{d^2}{dz^2} - K^2$

$W = \frac{\Gamma d^2}{\kappa} \hat{w} \qquad = D^2 - K^2$

$[\sigma - (D^2 - K^2)]\hat{T} = W$

$\left[\frac{\sigma}{Pr} - (D^2 - K^2)\right](D^2 - K^2)W = -R_a K^2 \hat{T}$

$R_a = \frac{g\alpha \Gamma d^4}{\kappa\nu}$ = Rayleigh Number

$W = DW = \hat{T} = 0 \quad z = \pm 1/2$

Buoyancy force = $g\alpha T = g\alpha \Gamma d$ = destabilizing

Viscous force = $\nu w / d^2 = \frac{\nu \kappa}{d^3}$ = stabilizing

$R_a = \frac{g\alpha\Gamma d}{\frac{\nu\kappa}{d^3}} = \frac{g\alpha\Gamma d^4}{\kappa\nu}$

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$\sigma_r = 0$ = marginal state

$\sigma_i = 0$: marginal state is stationary pattern of motion ie secondary flow superimposed on basic state "principle of exchange of stabilities"

$\sigma_i \neq 0$: marginal state is propagating waves in (x,y) directions

In general $\sigma_i \neq 0$, except Bernard and Taylor "type" problems of centrifugal instability. In these cases basic state transforms to another steady state. For general case, propagating waves either decay or grow in time.

For $Ra > 0$ ie top heavy condition it can be shown $\sigma_i = 0$!

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Two rigid plate Solution

For $\sigma_r = 0$:

$$(D^2 - K^2)\,\hat{T} = -W$$

$$(D^2 - K^2)^2 W = Ra\, K^2 \hat{T}$$

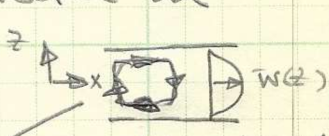
Combining:

$$(D^2 - K^2)^3 W = -Ra\, K^2 W$$

$$W = DW = (D^2 - K^2)^2 W = 0 \qquad z = \pm \tfrac{1}{2}$$

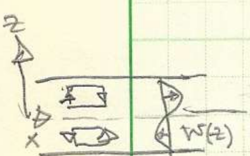
6^{th} order homogeneous ODE with 6 homogeneous BC

Eigenvalue problem, ie, non zero solutions only occur for particular value of Ra for each K. Note Pr not parameter in marginal state equations.



rising fluid looses heat by conduction to wall horz near wall then sinks

Solutions divided into symmetric vs. anti symmetric with respect to $z = 0$



Gravest / lowest modes are one row of cells (even) or two rows of cells (odd). Smallest critical Ra is for gravest even mode, as per EFD.

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General Solution

$$\bar{W}(z) = \sum_{i=1}^{6} e^{q_i z}$$

$\bar{W}$ into ODE : $(q_i^2 - K^2)^3 = -R_a K^2$ $\quad$ q_i corresponds to six roots of q equation

three roots q^2

$$\begin{cases} q^2 = -K^2\left[\left(\frac{R_a}{K^4}\right)^{1/3} - 1\right] \\ q^2 = K^2\left[1 + \frac{1}{2}\left(\frac{R_a}{K^4}\right)^{1/3}(1 \pm i\sqrt{3})\right] \end{cases}$$

$\sqrt{q^2}$ gives 6 roots: $\pm q_0$, $\pm q$, $\pm q^*$

$$q_0 = K\left[\left(\frac{R_a}{K^4}\right)^{1/3} - 1\right]^{1/2}$$

q, q^* roots of $\boxed{2}$

$$\bar{W}_{even} = A \cos q_0 z + B \cosh q z + C \cosh q^* z$$

BC:

$$D\bar{W} = -A q_0 \sin q_0 z + B q \sinh q z + C q^* \sinh q^* z$$

$$(D^2 - K^2)^2 \bar{W} = A(q_0^2 + K^2)^2 \cos q_0 z + B(q^2 - K^2)^2 \cosh q z + C(q^{*2} - K^2)^2 \cosh q^* z$$

$\bar{W} = D\bar{W} = (D^2 - K^2)\bar{W} = 0$ $\quad$ 3 equations 3 unknowns A, B, C

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$$\begin{bmatrix} \cos\frac{q_0}{2} & \cosh\frac{q}{2} & \cosh\frac{q^*}{2} \\ -q_0\sin\frac{q_0}{2} & q\sinh\frac{q}{2} & q^*\sinh\frac{q^*}{2} \\ (q_0^2+K^2)^2\cos\frac{q_0}{2} & (q^2-K^2)^2\cosh\frac{q}{2} & (q^{*2}-K^2)^2\cosh\frac{q^*}{2} \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = 0$$

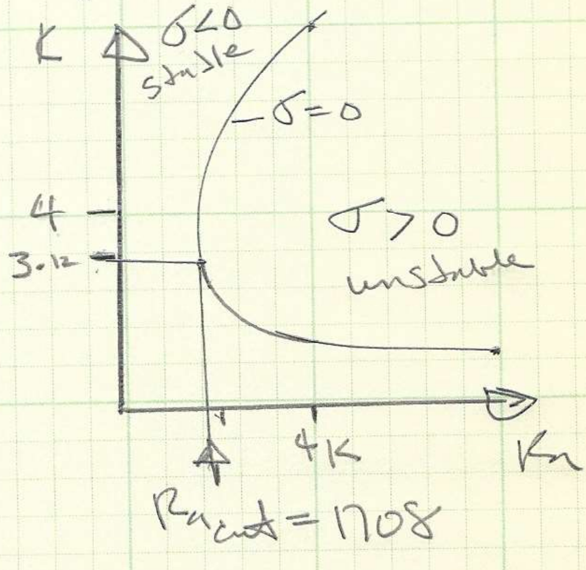
Non zero solution requires A, B, and C $\neq 0$, ie, determinant coefficient matrix $= 0$, which results in equation for

$$Ra = Ra(K)$$

K = eigenvalues

$K_{cr} = 3.12$

$$\lambda_{cr} = \frac{2\pi d}{K_{cr}} \sim 2d$$



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Two stress-free surface solution

Approximately realized using lighter fluid floating over heavier fluid. Main interest in simple solution.

$$w = T = \underbrace{\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) = \mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)} = 0 \qquad z = \pm 1/2$$

since $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0 \quad z = \pm 1/2$

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \quad z = \pm 1/2$$

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) = 0 \Rightarrow \frac{\partial^2 w}{\partial z^2} = 0$$

ODE: $(D^2-K^2)^3 W = -R_a K^2 W$

BC: $W = (D^2-K^2)^2 W = D^2 W = 0 \qquad z = \pm 1/2$

or $W = D^4 W = D^2 W = 0$

All even derivatives of W vanish on $z = \pm 1/2$ from ODE and BC:

$$W = A \sin n\pi z \qquad n = \text{integer}$$

Substitution into ODE:

$$Ra = (n^2\pi^2 + K^2)^3 / K^2$$

$$= \text{marginal state } Ra(K)$$

Gravest mode is for $n = 1$ and Ra_{crit} for

$$\frac{dRa}{dK^2} = 0 = \frac{3(\pi^2+K^2)^2}{K^2} - \frac{(\pi^2+K^2)^3}{K^4}$$

$$K^2_{crit} = \frac{\pi^2}{2}$$

$$Ra_{crit} = \frac{27}{4}\pi^4 = 657$$

<u>lower rigid plate / upper stress-free surface solution</u>

$$K_{crit} = 2.68$$

$$Ra_{crit} = 1101$$

most visual

Bernard 1900

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Cell Patterns

Linear theory gives Kcrit, but not cell pattern. K can be decomposed into infinite number of two orthogonal components. For isotropic horizonal EFD:
regular polygons in form of equilateral triangles, squares, regular hexagons all possible.

Ra_{crit} = regular hexagons

$Ra > Ra_{crit}$ = square convection rolls

$Ra >> Ra_{crit}$ = chaotic structure
$Ra = 5 \times 10^4$ = turbulent

magnitude & flow direction not predicted linear theory
Regular hexagon cells flow ~~in~~ center rises for liquid & falls for gases
attributed $\mu \downarrow$ liq & $\mu \uparrow$ gas for T $\uparrow$