Exercise 11.9. Consider the centrifugal instability problem of Section 11.6. From (11.51) and (11.53), the eigenvalue problem for determining the marginal state ($\sigma = 0$) is

$$\left(d^2/dR^2 - k^2\right)^2 \hat{u}_R = (1+\alpha x)\hat{u}_\varphi\,,\ \left(d^2/dR^2 - k^2\right)^2 \hat{u}_\varphi = -\mathrm{Ta}k^2\hat{u}_R\,, \qquad (11.92{,}93)$$

with $\hat{u}_R = d\hat{u}_R/dR = \hat{u}_\varphi = 0$ at $x = 0$ and 1. Conditions on $\hat{u}_\varphi$ are satisfied by assuming solutions of the form

$$\hat{u}_\varphi = \sum_{m=1}^{\infty} C_m \sin(m\pi x)\ . \qquad (11.94)$$

Inserting this into (11.92), obtain an equation for $\hat{u}_R$, and arrange so that the solution satisfies the four remaining conditions on $\hat{u}_R$. With $\hat{u}_R$ determined in this manner and $\hat{u}_\varphi$ given by (11.94), (11.93) leads to an eigenvalue problem for Ta(k). Following Chandrasekhar (1961, p. 300), show that the minimum Taylor number is given by (11.54) and is reached at $k_{cr} = 3.12$.

Solution 11.9. Continuing the effort from Exercise 11.8, insert (11.94) into (11.92) to find:

$$\left(d^2/dx^2 - K^2\right)^2 u = (1+\alpha x)v = (1+\alpha x)\sum_{m=1}^{\infty} C_m \sin(m\pi x)\,, \qquad (5)$$

where a switch has been made to the dimensionless variables of Exercise 11.8. Now arrange that the solution satisfies the four remaining boundary conditions on u. With u determined in this fashion and v given by (11.94), (11.93) will lead to an equation for Ta.

The solution of (5) is straightforward. The general solution can be written in the form:

$$u = \sum_{m=1}^{\infty} \frac{C_m}{(m^2\pi^2 + K^2)^2}\left\{\begin{array}{c} A_1^{(m)}\cosh Kx + B_1^{(m)}\sinh Kx + A_2^{(m)}x\cosh Kx + B_2^{(m)}x\sinh Kx \\ +\,(1+\alpha x)\sin(m\pi x) + \dfrac{4\alpha m\pi}{m^2\pi^2 + K^2}\cos(m\pi x) \end{array}\right\}, \qquad (6)$$

where the constants are determined by the boundary conditions $u = (d/dx)u = 0$ at $x = 0$ and 1. These conditions lead to the four equations:

$$A_1^{(m)} = -\frac{4\alpha m\pi}{m^2\pi^2 + K^2},\ KB_1^{(m)} + A_2^{(m)} = -m\pi,$$

$$A_1^{(m)}\cosh K + B_1^{(m)}\sinh K + A_2^{(m)}\cosh K + B_2^{(m)}\sinh K = (-1)^{m+1}\frac{4\alpha m\pi}{m^2\pi^2 + K^2}, \qquad (7)$$

$$A_1^{(m)}K\sinh K + B_1^{(m)}K\cosh K + A_2^{(m)}(\cosh K + K\sinh K) + B_2^{(m)}(\sinh K + K\cosh K) = (-1)^{m+1}(1+\alpha)m\pi$$

The solution of these equations is:

$$A_1^{(m)} = -\frac{4\alpha m\pi}{m^2\pi^2 + K^2}$$

$$B_1^{(m)} = \frac{m\pi}{\Delta}\left\{K + \beta_m(\sinh K + K\cosh K) - \gamma_m \sinh K\right\}$$

$$A_2^{(m)} = -\frac{m\pi}{\Delta}\left\{\sinh^2 K + \beta_m K(\sinh K + K\cosh K) - \gamma_m K\sinh K\right\}$$

$$B_2^{(m)} = \frac{m\pi}{\Delta}\left\{(\sinh K\cosh K - K) + \beta_m K^2\sinh K - \gamma_m(K\cosh K - \sinh K)\right\} \qquad (8)$$

where:

$$\Delta = \sinh^2 K - K^2,\quad \beta_m = \frac{4\alpha}{m^2\pi^2 + K^2}\left\{(-1)^{m+1} + \cosh K\right\},\quad \text{and}$$

$$\gamma_m = (-1)^{m+1}(1-\alpha) + \frac{4\alpha}{m^2\pi^2 + K^2}K\sinh K\,.$$

Substituting u from (6) and v from (11.94) into (11.93) produces:

$$\sum_{n=1}^{\infty} C_n\left(n^2\pi^2+K^2\right)\sin n\pi x$$

$$= TaK^2\sum_{m=1}^{\infty}\frac{C_m}{(m^2\pi^2+K^2)^2}\left\{\begin{array}{c} A_1^{(m)}\cosh Kx + B_1^{(m)}\sinh Kx + A_2^{(m)}x\cosh Kx + B_2^{(m)}x\sinh Kx \\ +(1+\alpha x)\sin(m\pi x) + \dfrac{4\alpha m\pi}{m^2\pi^2+K^2}\cos(m\pi x)\end{array}\right\}. \quad (9)$$

Multiply this equation by $\sin(n\pi x)$ and integrate from $x = 0$ to $x = 1$, to obtain a system of linear homogeneous equations for the constants C_m. The requirement that these constants are not all zero leads to the equation:

$$\frac{n\pi}{n^2\pi^2+K^2}\left\{\begin{array}{l}\left[1+(-1)^{n+1}\cosh K\right]A_1^{(m)}+\left[(-1)^{n+1}\sinh K\right]B_1^{(m)}+(-1)^{n+1}\left[\cosh K-\dfrac{2K}{n^2\pi^2+K^2}\sinh K\right]A_2^{(m)} \\ +\left[(-1)^{n+1}\sinh K-\dfrac{2K}{n^2\pi^2+K^2}\left\{1+(-1)^{n+1}\cosh K\right\}\right]B_2^{(m)}\end{array}\right\}$$

$$+\alpha x_{nm}+\frac{1}{2}\delta_{nm}-\frac{1}{2}\left(n^2\pi^2+K^2\right)^3\frac{\delta_{nm}}{K^2Ta}=0 \quad (10)$$

where

$$x_{nm}=\left\{\begin{array}{ll} 0 & if\ m+n\ is\ even\ and\ m\neq n \\ 1/4 & if\ m=n \\ \dfrac{4nm}{n^2-m^2}\left\{\dfrac{2}{m^2\pi^2+K^2}-\dfrac{1}{\pi^2(n^2-m^2)}\right\} & if\ m+n\ is\ odd\end{array}\right\}.$$

On using the first two equations of (7), equation (10) simplifies to:

$$\frac{n\pi}{n^2\pi^2+K^2}\left\{\frac{4m\pi\alpha}{m^2\pi^2+K^2}\left[(-1)^{m+n}-1\right]-\frac{2K}{n^2\pi^2+K^2}\left[(-1)^{n+1}A_2^{(m)}\sinh K+(-1)^{n+1}B_2^{(m)}\cosh K+B_2^{(m)}\right]\right\}$$

$$+\alpha x_{nm}+\frac{1}{2}\delta_{nm}-\frac{1}{2}\left(n^2\pi^2+K^2\right)^3\frac{\delta_{nm}}{K^2Ta}=0 \quad (11)$$

After substituting for the constants $A_2^{(m)}$ and $B_2^{(m)}$ given by (8), (11) becomes:

$$\frac{4mn\pi^2\alpha}{\left(n^2\pi^2+K^2\right)\left(m^2\pi^2+K^2\right)}\left[(-1)^{m+n}-1\right]$$

$$-\frac{2Kmn\pi^2}{\left(n^2\pi^2+K^2\right)\left(\sinh^2K-K^2\right)}\left\{\begin{array}{l}(\sinh K\cosh K-K)\left[1+(1+\alpha)(-1)^{m+n}\right] \\ +(\sinh K-K\cosh K)\left[(-1)^{n+1}+(1+\alpha)(-1)^{m+1}\right] \\ -\dfrac{4k\alpha\sinh K}{m^2\pi^2+K^2}\left[\sinh K+K(-1)^{m+1}\right]\left[(-1)^{m+n}-1\right]\end{array}\right\}$$

$$+\alpha x_{nm}+\frac{1}{2}\delta_{nm}-\frac{1}{2}\left(n^2\pi^2+K^2\right)^3\frac{\delta_{nm}}{K^2Ta}=0 \quad (12)$$

A first approximation to the solution of (12) is obtained by setting the (1,1)-element of the determinant zero. This implies:

$$\frac{1}{2}\left(\pi^2+K^2\right)^3\frac{\delta_{nm}}{K^2Ta}=\frac{1}{4}\alpha+\frac{1}{2}-\frac{2K\pi^2(2+\alpha)}{\left(\pi^2+K^2\right)\left(\sinh^2K-K^2\right)}\left[(\sinh K\cosh K-K)+(\sinh K-K\cosh K)\right]$$

and this simplifies to:

$$Ta=\frac{2}{2+\alpha}\frac{\left(\pi^2+K^2\right)^3}{K^2\left\{1-16K\pi^2\cosh^2(K/2)\Big/\left[\left(\pi^2+K^2\right)^2(\sinh K+K)\right]\right\}}$$

A plot of $Ta(2+\alpha)$ as a function of K from this solution shows that the minimum value of the Taylor number is:

$$Ta_{cr}=3430/(2+\alpha),\ \text{and is reached at } K_{cr}=3.12.$$