Exercise 11.8. Consider the centrifugal instability problem of Section 11.6. Making the narrow-gap approximation, work out the algebra of going from (11.50) to (11.51).

Solution 11.8. The perturbation equations (11.50) are:

$$\frac{\partial u_R}{\partial t}-\frac{2U_\varphi u_\varphi}{R}=-\frac{1}{\rho}\frac{\partial p}{\partial R}+\nu\left(\nabla^2 u_R-\frac{u_R}{R^2}\right),\quad \frac{\partial u_\varphi}{\partial t}+\left(\frac{dU_\varphi}{dR}+\frac{U_\varphi}{R}\right)u_R=\nu\left(\nabla^2 u_\varphi-\frac{u_\varphi}{R^2}\right),$$

$$\frac{\partial u_z}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 u_z,\ \text{and}\ \frac{\partial}{\partial R}\left(Ru_R\right)+\frac{\partial u_z}{\partial z}=0. \tag{1}$$

Substituting in

$$u_R=u(R)e^{\sigma t}\cos kz,\ u_\varphi=v(R)e^{\sigma t}\cos kz,\ u_z=w(R)e^{\sigma t}\sin kz\ ,\text{and}\ p/\rho=\hat{p}(R)e^{\sigma t}\cos kz,$$

the equation set (1) becomes

$$\nu\left(\frac{d}{dR}\left(\frac{d}{dR}+\frac{1}{R}\right)-k^2-\frac{\sigma}{\nu}\right)u+2\frac{U_\varphi}{R}v=\frac{d\hat{p}}{dR},$$

$$\nu\left(\frac{d}{dR}\left(\frac{d}{dR}+\frac{1}{R}\right)-k^2-\frac{\sigma}{\nu}\right)v-\left(\frac{dU_\varphi}{dR}+\frac{U_\varphi}{R}\right)u=0,$$

$$\nu\left(\left(\frac{d}{dR}+\frac{1}{R}\right)\frac{d}{dR}-k^2-\frac{\sigma}{\nu}\right)w=-k\hat{p}\ ,\ \text{and}$$

$$\left(\frac{d}{dR}+\frac{1}{R}\right)u=-kw. \tag{2}$$

Eliminating w between the third and fourth equations of set (2) produces:

$$\frac{\nu}{k^2}\left(\left(\frac{d}{dR}+\frac{1}{R}\right)\frac{d}{dR}-k^2-\frac{\sigma}{\nu}\right)\left(\frac{d}{dR}+\frac{1}{R}\right)u=\hat{p}.$$

Inserting this equation for $\hat{p}$ into the first equation of set (2), and working on the algebra eventually leads to:

$$\frac{\nu}{k^2}\left(\frac{d}{dR}\left(\frac{d}{dR}+\frac{1}{R}\right)-k^2-\frac{\sigma}{\nu}\right)\left(\frac{d}{dR}\left(\frac{d}{dR}+\frac{1}{R}\right)-k^2\right)u=2\frac{U_\varphi}{R}v. \tag{3}$$

The second equation of set (2) and equation (3) are a pair of equations relating u and v.

Using the radius of the outer cylinder R_2, define new dimensionless variables and parameters:

$$r=R/R_2,\ k^2=K^2/R_2^2,\ \text{and}\quad \omega=\sigma R_2^2/\nu,$$

so that the relevant equation pair becomes:

$$\left(\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)-K^2-\omega\right)\left(\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)-K^2\right)u=2\frac{BK^2}{\nu}\left(\frac{1}{r^2}+\frac{AR_2^2}{B}\right)v,\ \text{and}$$

$$\left(\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)-K^2-\omega\right)v=2\frac{A}{\nu}R_2^2u,$$

where $U_\varphi = Ar + B/r$. It is convenient to make the transformation

$$2\frac{AR_2^2}{\nu}u\rightarrow u,$$

so that the equations take the more convenient forms:

$$\left(\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)-K^2-\omega\right)\left(\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)-K^2\right)u=-TaK^2\left(\frac{1}{r^2}-\kappa\right)v,\ \text{and}$$

$$\left(\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)-K^2-\omega\right)v=u,$$

where $Ta=-4\frac{AB}{\nu^2}R_2^2=4\frac{\Omega_1^2R_1^4}{\nu^2}\frac{(1-\mu)\left(1-\mu/\eta^2\right)}{\left(1-\eta^2\right)^2}$, $\kappa=-\frac{A}{B}R_2^2=\frac{1-\mu/\eta^2}{1-\mu}$, $\mu=\frac{\Omega_2}{\Omega_1}$, $\eta=\frac{R_1}{R_2}$,

$A=-\Omega_1\eta^2\frac{1-\mu/\eta^2}{1-\eta^2}$, and $B=\Omega_1R_1^2\frac{1-\mu}{1-\eta^2}$.

The no-slip and boundary conditions at the walls require:

$$u=v=0 \text{ and } (d/dr)u=0 \text{ at } r=\eta \text{ and } 1,$$

where the last of the three conditions is equivalent to $w=0$ (see the final equation of set (1)).

Now consider the narrow gap approximation that is valid when

$$R_2-R_1<<\tfrac{1}{2}(R_2+R_1).$$

When this is true, $d/dr>>1/r$ so

$$\frac{d}{dr}+\frac{1}{r}\cong\frac{d}{dr}, \text{ and } A+\frac{B}{r^2}\cong\Omega_1\left[1-(1-\mu)\frac{r-R_1}{R_2-R_1}\right].$$

Now convert the independent radial coordinate (R) to one (x) that starts on the inner cylinder using the gap dimension, $d=R_2-R_1$, as the length scale, and let $k=K/d$, and $w=\omega d^2/\nu$ to find:

$$\left(\frac{d^2}{dx^2}-K^2-\omega\right)\left(\frac{d^2}{dx^2}-K^2\right)u=\frac{2\Omega_1d^4}{\nu}K^2(1-(1-\mu)x)v, \text{ and}$$

$$\left(\frac{d^2}{dx^2}-K^2-\omega\right)v=\frac{2Ad^4}{\nu}u,$$

as the relevant equation set. By the further transformation $u\to\frac{2\Omega_1d^2K^4}{\nu}u$, these equations become:

$$\left(\frac{d^2}{dx^2}-K^2-\omega\right)\left(\frac{d^2}{dx^2}-K^2\right)u=(1+\alpha x)v, \text{ and}$$

$$\left(\frac{d^2}{dx^2}-K^2-\omega\right)v=-TaK^2u,$$

where

$$Ta=-\frac{4A\Omega_1}{\nu^2}d^4, \text{ and } \alpha=-(1-\mu).$$

These are the same equations as (11.51).