Chapter 4: Additional Topics in Boundary Layer Theory

Axisymmetric, Three-Dimensional, and Homann Flow

Fluid Mechanics — Advanced Topics

Contents

1 Axisymmetric Boundary Layers

We consider axisymmetric flows with no swirl, where the boundary layer thickness δ is small compared to both the longitudinal and transverse radii of curvature. The governing equations were derived by Mangler (1945) and differ from the two-dimensional planar case through the appearance of the local body radius $r_0(x)$ in the continuity equation:

$$\frac{\partial}{\partial x}(r_0 u) + r_0 \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$0 = \frac{\partial p}{\partial y} \tag{3}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u_e \frac{du_e}{dx} \tag{4}$$

Here x is measured along the body surface, y is the wall-normal coordinate, u and v are the corresponding velocity components, and $u_e(x)$ is the inviscid edge velocity.

1.1 Mangler's Transformation

Mangler (1945) showed that the axisymmetric boundary layer equations can be mapped exactly onto the two-dimensional planar equations via the coordinate transformation

$$x' = \int_0^x \left(\frac{r_0}{L}\right)^2 dx, \qquad y' = \frac{r_0\, y}{L}, \tag{5}$$

$$u' = u, \qquad U'(x') = U(x), \tag{6}$$

$$v' = \frac{L}{r_0}\left(v + \frac{y\,u}{r_0}\frac{dr_0}{dx}\right), \tag{7}$$

where L is an arbitrary reference length. The Jacobian entries needed for substitution are

$$\frac{\partial x'}{\partial x} = \left(\frac{r_0}{L}\right)^2, \qquad \frac{\partial y'}{\partial x} = \frac{y}{L}\frac{\partial r_0}{\partial x}.$$

Substitution into Eqs. (1)–(4) produces the standard planar boundary layer equations:

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0, \tag{8}$$

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = U'\frac{\partial U'}{\partial x'} + \nu \frac{\partial^2 u'}{\partial y'^2}. \tag{9}$$

When $r_0 = \text{const}$ the transformation is trivially the identity.

2 Homann Flow: Axisymmetric Stagnation Point

The axisymmetric stagnation point on a blunt body is a classical and important special case of Mangler's transformation. The corresponding self-similar flow is known as *Homann flow* (Homann 1936).

2.1 Physical Setup and Inviscid Edge Velocity

Near a three-dimensional (axisymmetric) stagnation point the inviscid (irrotational) flow solution gives an edge velocity that grows linearly with the arc length x from the stagnation point:

$$\frac{u_e}{u_0} = \frac{x}{L}, \qquad u_0 = \alpha\, u_\infty, \tag{10}$$

where L is a characteristic body dimension and α depends on the body shape. This is the axisymmetric analogue of the planar Hiemenz stagnation-point velocity.

For a sphere of radius R the inviscid solution gives $\alpha = 3/2$. For a general blunt body α encodes the local curvature at the stagnation point.

The body surface near the stagnation point is approximately a paraboloid of revolution; to leading order $r_0 \approx x$, i.e.

$$r_0(x) = x. \tag{11}$$

2.2 Applying Mangler's Transformation

Derivation 2.1: Transformed coordinates for Homann flow

Substituting $r_0 = x$ into Eq. (5):

$$\frac{x'}{L} = \int_0^{x/L} \left(\frac{r_0}{L}\right)^2 d\left(\frac{x}{L}\right) = \int_0^{x/L} \left(\frac{x}{L}\right)^2 d\left(\frac{x}{L}\right) = \frac{1}{3}\left(\frac{x}{L}\right)^3. \tag{12}$$

$$\frac{y'}{L} = \frac{r_0}{L} \cdot \frac{y}{L} = \frac{x}{L} \cdot \frac{y}{L}. \tag{13}$$

These define the equivalent plane-flow point (x', y') corresponding to the axisymmetric point (x, y).

2.3 Equivalent Plane-Flow Edge Velocity

Derivation 2.2: Transformed edge velocity

The edge velocity transforms as $u_e' = u_e$. Using Eq. (10) and inverting Eq. (12):

$$\frac{x}{L} = \left(3\,\frac{x'}{L}\right)^{1/3},$$

so

$$\frac{u_e'}{u_0} = \frac{u_e}{u_0} = \frac{x}{L} = \left(3\,\frac{x'}{L}\right)^{1/3} \propto x'^{1/3}. \tag{14}$$

The reference velocity constant is unchanged; only the length scale shifts to $L' = L^3$. The equivalent plane flow therefore has $u_e' \propto x'^{1/3}$, i.e. it is a *Falkner–Skan* flow with exponent $m = 1/3$ (corresponding to a 45° wedge, $\beta = 2m/(m+1) = 1/2$).

2.4 Falkner–Skan Connection

The Falkner–Skan family of similarity solutions applies to power-law edge velocities $u_e \propto x^m$. The similarity variable and stream function are

$$\eta = y\sqrt{\frac{(m+1)U_e}{2\nu x}}, \qquad \psi = \sqrt{\frac{2\nu x\, U_e}{m+1}}\, f(\eta), \tag{15}$$

where $f(\eta)$ satisfies the Falkner–Skan ODE:

$$f''' + f\, f'' + \beta\big(1 - f'^2\big) = 0, \qquad \beta = \frac{2m}{m+1}. \tag{16}$$

Boundary conditions: $f(0) = f'(0) = 0$ (no-slip, no penetration) and $f'(\infty) = 1$ (matching to edge velocity).

For Homann flow the equivalent plane problem has $m = 1/3$, giving $\beta = 1/2$, and Eq. (16) becomes:

$$\boxed{f''' + f\, f'' + \frac{1}{2}\big(1 - f'^2\big) = 0.} \tag{17}$$

This ODE must be solved numerically (shooting method). The key surface quantities are $f''(0) \approx 0.9277$ (compared to $f''(0) = 1.2326$ for the planar Hiemenz case, $m = 1$).

2.5 Velocity Field in the Original Coordinates

Once the Falkner–Skan solution $f'(\eta)$ is known, the u-velocity in the original axisymmetric coordinates (x, y) is recovered via

$$\frac{u(x,y)}{u_0} = f'\left[\eta\left(\frac{x}{L}\cdot\frac{y}{L},\ \frac{1}{3}\left(\frac{x}{L}\right)^3\right)\right]. \tag{18}$$

The v-component follows by inverting the Mangler v-transformation, Eq. (7):

$$v = \frac{r_0}{L}\, v' - \frac{y\, u}{r_0}\frac{dr_0}{dx} = \frac{x}{L}\, v' - \frac{y\, u}{x}. \tag{19}$$

2.6 Boundary Layer Thickness Comparison

A key result from the Homann solution is:

Key Result 2.1: Axisymmetric vs. planar stagnation point thickness

The boundary layer at an **axisymmetric** stagnation point is approximately **80%** as thick as at the equivalent **planar** stagnation point:

$$\frac{\delta_{\text{axi}}}{\delta_{\text{plane}}} \approx 0.80.$$

Physically, the azimuthal stretching in the axisymmetric geometry provides an additional strain rate that thins the boundary layer relative to the 2D case.

The displacement thickness for the Homann flow scales as

$$\delta^* = \sqrt{\frac{\nu}{\alpha\, u_\infty/L}}\, \delta^*_{\text{FS}}(m = 1/3), \tag{20}$$

where δ^*_{FS} is the non-dimensional displacement thickness from the Falkner–Skan solution.

2.7 Extension to Flow over a Cone

The axisymmetric stagnation point is a special case of streaming flow over a cone of half-angle $\theta_{1/2}$. For a cone:

- External velocity: $u_e \propto x^n$ (power-law, where n depends on cone angle).
- Surface radius: $r_0 = x \sin\theta_{1/2}$.

Applying Mangler's transformation with these gives

$$u_e' \propto x'^{n/3}, \tag{21}$$

which is again a Falkner–Skan solution with effective exponent $m = n/3$. Unfortunately, there is no simple closed-form relation between the cone angle $\theta_{1/2}$ and the exponent n; Whitehead & Canetti (1950) provide a graphical relationship.

3 Three-Dimensional Boundary Layers

When the flow is driven by pressure gradients in both streamwise (x) and spanwise (z) directions, three velocity components appear and the boundary layer equations become:

$$u_x + v_y + w_z = 0, \tag{22}$$

$$u\, u_x + v\, u_y + w\, u_z = -\frac{p_x}{\rho} + \nu\, u_{yy}, \tag{23}$$

$$u\, w_x + v\, w_y + w\, w_z = -\frac{p_z}{\rho} + \nu\, w_{yy}. \tag{24}$$

Complex crossflow patterns (bidirectional skewing in hodograph space) exhausted the capability of 3D integral methods, motivating the development of 3D differential boundary layer methods (Nash & Patel 1972). Thick boundary layers and separation then drove rapid extension of differential methods into modern CFD: RANS, HRLES, LES, and DNS.

3.1 Flat Plate BL with Constant Transverse Pressure Gradient

Consider a flat plate with leading edge at $x = 0$ and inflow at angle θ_0. The boundary conditions as $y \to \infty$ are:

$$u = u_e = \text{const}, \qquad p_x = 0, \tag{25}$$

$$w = w_e = u_e(a + bx), \qquad -\frac{p_z}{\rho} = u_e\, w_{e,x} = u_e^2\, b, \tag{26}$$

where b measures the magnitude of the transverse pressure gradient p_z. The vorticity of the external w-field is

$$w_y = -\frac{\partial w}{\partial x} = -u_e b.$$

For $b = 0$, $\theta = \theta_0 = \tan^{-1}(w_e/u_e) = \tan^{-1} a$, which recovers the effect of sweep on a Blasius boundary layer. The external streamline trajectory is parabolic:

$$\frac{dz_e}{dx} = \frac{w_e}{u_e} = a + bx \quad \Longrightarrow \quad z_e = ax + \tfrac{1}{2}bx^2 + c, \quad c = z_e\big|_{\text{LE}}.$$

Sweep Independence Principle

Because $p_x = 0$ the continuity and x-momentum equations decouple from w:

$$u_x + v_y = 0, \qquad u\, u_x + v\, u_y = \nu\, u_{yy}.$$

This is precisely the Blasius flat-plate problem, giving

$$\frac{u}{u_e} = f'(\eta), \qquad \eta = y\sqrt{\frac{u_e}{\nu x}}.$$

The w-momentum equation is then forced by the known u, v field and admits the decomposition

$$\frac{w}{u_e} = \frac{w_e(x)}{u_e} f'(\eta) + bx\, h(\eta), \tag{27}$$

where the first term is the Blasius component scaled to match $w_e(x)$, and $h(\eta)$ satisfies

$$h'' + \tfrac{1}{2} f\, h' - f'\, h + 1 - f'^2 = 0, \qquad h(0) = h(\infty) = 0. \tag{28}$$

Equation (28) is linear and solved numerically, analogous to the Blasius equation itself.

Velocity Profiles in Surface-Aligned Coordinates

Transforming to normal (n) and tangential (t) coordinates aligned with the local outer flow direction and normalising by the outer speed $V_\infty = [u_e^2 + w_e^2]^{1/2}$:

$$\frac{v_t}{V_\infty} = f'(\eta) + \tfrac{1}{2}\sin 2\theta\,(\tan\theta - \tan\theta_0)\,h(\eta), \tag{29}$$

$$\frac{v_n}{V_\infty} = \tfrac{1}{2}\cos^2\theta\,(\tan\theta - \tan\theta_0)\,h(\eta), \tag{30}$$

where $\theta = \tan^{-1}(w_e/u_e)$ is the local inviscid streamline angle.

Notable features of the solution include: (i) $v_t < 0$ near the wall but no actual flow reversal; (ii) an overshoot in v_t in the outer boundary layer, analogous to Stokes' second problem for an oscillating outer flow, where the net transverse viscous force opposes the pressure force and decays at large η; and (iii) the flow **never actually separates**. The surface streamline trajectory in the z-direction is also parabolic:

$$z_s = ax + \tfrac{1}{2}bx^2\left[1 + \frac{h'(0)}{f''(0)}\right] + C_0, \qquad 1 + \frac{h'(0)}{f''(0)} \approx 1 + \tfrac{1}{3},$$

differing from the external streamline z_e by the factor $1 + 1/3$.

Summary of Key Results

Flow	**Method**	**Key result**
Axisymmetric BL	Mangler transform	Reduces to planar BL
Homann (axi. stag.)	FS with $m = 1/3$	$\delta_{\text{axi}}/\delta_{\text{plane}} \approx 0.80$
Cone flow	Mangler transform	FS with $m = n/3$
3D flat-plate BL	Sweep independence	u, v decouple from w