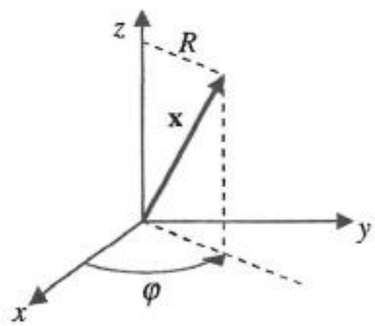
Cylindrical Coordinates (Figure B.2)



Position: $\mathbf{x} = (R, \varphi, z) = R\mathbf{e}_R + z\mathbf{e}_z$; $x = R\cos\varphi$, $y = R\sin\varphi$, $z = z$; or $R = \sqrt{x^2 + y^2}$, $\varphi = \tan^{-1}(y/x)$

Unit vectors: $\mathbf{e}_R = \mathbf{e}_x \cos\varphi + \mathbf{e}_y \sin\varphi$, $\mathbf{e}_\varphi = -\mathbf{e}_x \sin\varphi + \mathbf{e}_y \cos\varphi$, $\mathbf{e}_z$ = same as Cartesian

Unit vector dependencies: $\partial \mathbf{e}_R/\partial R = 0$, $\partial \mathbf{e}_R/\partial \varphi = \mathbf{e}_\varphi$, $\partial \mathbf{e}_R/\partial z = 0$

$$\partial \mathbf{e}_\varphi/\partial R = 0,\ \partial \mathbf{e}_\varphi/\partial \varphi = -\mathbf{e}_R,\ \partial \mathbf{e}_\varphi/\partial z = 0$$

$$\partial \mathbf{e}_z/\partial R = 0,\ \partial \mathbf{e}_z/\partial \varphi = 0,\ \partial \mathbf{e}_z/\partial z = 0$$

Gradient Operator: $\nabla = \mathbf{e}_R \dfrac{\partial}{\partial R} + \mathbf{e}_\varphi \dfrac{1}{R}\dfrac{\partial}{\partial \varphi} + \mathbf{e}_z \dfrac{\partial}{\partial z}$

Surface integral, S, of $f(R, \theta, z)$ over the cylinder defined by

$$R = \xi: S = \int_{\varphi=0}^{2\pi} \int_{z=-\infty}^{+\infty} f(\xi, \varphi, z)\xi \, dz \, d\varphi$$

Surface integral, S, of $f(R, \theta, z)$ over the half plane defined by

$$\varphi = \psi: S = \int_{R=0}^{+\infty} \int_{z=-\infty}^{+\infty} f(R, \psi, z) dz \, dR$$

Surface integral, S, of $f(R, \theta, z)$ over the plane defined by $z = \zeta$: $S = \displaystyle\int_{R=0}^{+\infty} \int_{\varphi=0}^{2\pi} f(R, \varphi, \zeta) R \, d\varphi \, dR$

Volume integral, V, of $f(R, \theta, z)$ over all space: $V = \displaystyle\int_{z=-\infty}^{+\infty} \int_{R=0}^{+\infty} \int_{\varphi=0}^{2\pi} f(R, \varphi, z) R \, d\varphi \, dR \, dz$

Cylindrical Coordinates (Figure B.2)

Position and velocity vectors: $\mathbf{x} = (R, \varphi, z) = R\mathbf{e}_R + z\mathbf{e}_z$; $\mathbf{u} = (u_R, u_\varphi, u_z) = u_R\mathbf{e}_R + u_\varphi\mathbf{e}_\varphi + u_z\mathbf{e}_z$

Gradient of a scalar ψ: $\nabla\psi = \mathbf{e}_R\dfrac{\partial\psi}{\partial R} + \mathbf{e}_\varphi\dfrac{1}{R}\dfrac{\partial\psi}{\partial\varphi} + \mathbf{e}_z\dfrac{\partial\psi}{\partial z}$

Laplacian of a scalar ψ: $\nabla^2\psi = \dfrac{1}{R}\dfrac{\partial}{\partial R}\left(R\dfrac{\partial\psi}{\partial R}\right) + \dfrac{1}{R^2}\dfrac{\partial^2\psi}{\partial\varphi^2} + \dfrac{\partial^2\psi}{\partial z^2}$

Divergence of a vector: $\nabla\cdot\mathbf{u} = \dfrac{1}{R}\dfrac{\partial}{\partial R}(Ru_R) + \dfrac{1}{R}\dfrac{\partial u_\varphi}{\partial\varphi} + \dfrac{\partial u_z}{\partial z}$

Curl of a vector, vorticity: $\boldsymbol{\omega} = \nabla\times\mathbf{u} = \mathbf{e}_R\left(\dfrac{1}{R}\dfrac{\partial u_z}{\partial\varphi} - \dfrac{\partial u_\varphi}{\partial z}\right) + \mathbf{e}_\varphi\left(\dfrac{\partial u_R}{\partial z} - \dfrac{\partial u_z}{\partial R}\right) + \mathbf{e}_z\left(\dfrac{1}{R}\dfrac{\partial(Ru_\varphi)}{\partial R} - \dfrac{1}{R}\dfrac{\partial u_R}{\partial\varphi}\right)$

Laplacian of a vector: $\nabla^2\mathbf{u} = \mathbf{e}_R\left(\nabla^2 u_R - \dfrac{u_R}{R^2} - \dfrac{2}{R^2}\dfrac{\partial u_\varphi}{\partial\varphi}\right) + \mathbf{e}_\varphi\left(\nabla^2 u_\varphi + \dfrac{2}{R^2}\dfrac{\partial u_R}{\partial\varphi} - \dfrac{u_\varphi}{R^2}\right) + \mathbf{e}_z\nabla^2 u_z$

Strain rate S_{ij} and viscous stress τ_{ij} for an incompressible fluid where $\tau_{ij} = 2\mu S_{ij}$:

$$S_{RR} = \frac{\partial u_R}{\partial R} = \frac{1}{2\mu}\tau_{RR}, \quad S_{\varphi\varphi} = \frac{1}{R}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_R}{R} = \frac{1}{2\mu}\tau_{\varphi\varphi}, \quad S_{zz} = \frac{\partial u_z}{\partial z} = \frac{1}{2\mu}\tau_{zz}$$

$$S_{R\varphi} = \frac{R}{2}\frac{\partial}{\partial R}\left(\frac{u_\varphi}{R}\right) + \frac{1}{2R}\frac{\partial u_R}{\partial\varphi} = \frac{1}{2\mu}\tau_{R\varphi}, \quad S_{\varphi z} = \frac{1}{2R}\frac{\partial u_z}{\partial\varphi} + \frac{1}{2}\frac{\partial u_\varphi}{\partial z} = \frac{1}{2\mu}\tau_{\varphi z},$$

$$S_{zR} = \frac{1}{2}\left(\frac{\partial u_R}{\partial z} + \frac{\partial u_z}{\partial R}\right) = \frac{1}{2\mu}\tau_{zR}$$

Equation of continuity: $\dfrac{\partial\rho}{\partial t} + \dfrac{1}{R}\dfrac{\partial}{\partial R}(R\rho u_R) + \dfrac{1}{R}\dfrac{\partial}{\partial\varphi}(\rho u_\varphi) + \dfrac{\partial}{\partial z}(\rho u_z) = 0$

Navier-Stokes equations with constant ρ, constant ν, and no body force:

$$\frac{\partial u_R}{\partial t} + (\mathbf{u}\cdot\nabla)u_R - \frac{u_\varphi^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + \nu\left(\nabla^2 u_R - \frac{u_R}{R^2} - \frac{2}{R^2}\frac{\partial u_\varphi}{\partial\varphi}\right),$$

$$\frac{\partial u_\varphi}{\partial t} + (\mathbf{u}\cdot\nabla)u_\varphi + \frac{u_R u_\varphi}{R} = -\frac{1}{\rho R}\frac{\partial p}{\partial\varphi} + \nu\left(\nabla^2 u_\varphi + \frac{2}{R^2}\frac{\partial u_R}{\partial\varphi} - \frac{u_\varphi}{R^2}\right),$$

$$\frac{\partial u_z}{\partial t} + (\mathbf{u}\cdot\nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z$$

where: $\mathbf{u}\cdot\nabla = u_R\dfrac{\partial}{\partial R} + \dfrac{u_\varphi}{R}\dfrac{\partial}{\partial\varphi} + u_z\dfrac{\partial}{\partial z}$ and $\nabla^2 = \dfrac{1}{R}\dfrac{\partial}{\partial R}\left(R\dfrac{\partial}{\partial R}\right) + \dfrac{1}{R^2}\dfrac{\partial^2}{\partial\varphi^2} + \dfrac{\partial^2}{\partial z^2}$.

Steady Flow Between Concentric Rotating Cylinders

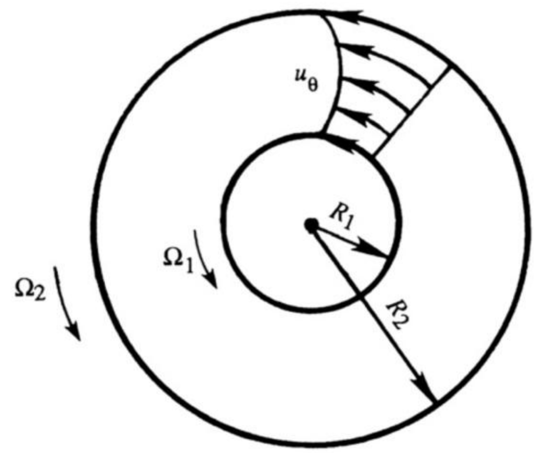


FIGURE 8.6 Circular Couette flow. The viscous fluid flows in the gap between an inner cylinder with radius R_1 that rotates at angular speed Ω_1 and an outer cylinder with radius R_2 that rotates at angular speed Ω_2.

$\underline{u} = (0, u_\theta(R), 0)$

$$R^{-1}\cancel{\frac{\partial}{\partial R}(Ru_R)} + R^{-1}\cancel{\frac{\partial}{\partial \theta}(u_\theta)} + \cancel{\frac{\partial}{\partial z}(u_z)} = 0$$

Continuity automatically satisfied

R-momentum: $-\frac{u_\theta^2}{R} = -\frac{1}{\rho}\frac{dp}{dR} \quad p(R) = f\big(u_\theta(R)\big)$

θ-momentum: $0 = \mu\frac{d}{dR}\left[R^{-1}\frac{d}{dR}(Ru_\theta)\right]$

double integration ⇒

$$u_\theta(R) = AR + \frac{B}{R} \qquad u_\theta(R_1) = \Omega_1 R_1, \quad u_\theta(R_2) = \Omega_2 R_2$$

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \qquad B = \frac{(\Omega_2 - \Omega_1)R_1^2 R_2^2}{R_1^2 - R_2^2}$$

$$u_\theta(R) = \frac{1}{R_2^2 - R_1^2}\left[(\Omega_2 R_2^2 - \Omega_1 R_1^2)R - \frac{(\Omega_2 - \Omega_1)R_1^2 R_2^2}{R}\right]$$

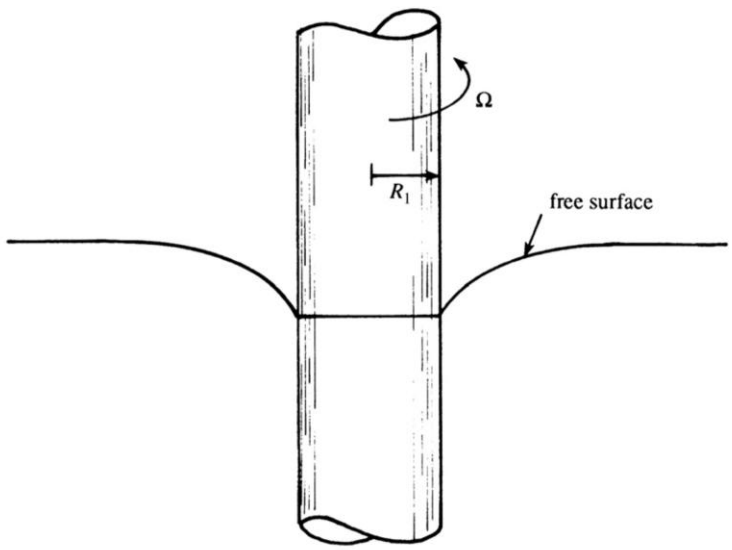
Interesting limiting cases:

(1) $R_2 \to \infty \quad \Omega_2 = 0$

(2) $R_1 \to 0 \quad \Omega_1 = 0$

(1)

FIGURE 8.7 Rotation of a solid cylinder of radius R_1 in an infinite body of viscous fluid. If gravity points downward along the cylinder's axis, the shape of a free surface pierced by the cylinder is also indicated. The flow field is viscous but irrotational.



$u_\theta(R) = \frac{\Omega_1 R_1^2}{R}$ Same as ideal vortex for $R > R_1$ $\quad \Gamma = 2\pi\Omega_1 R_1^2$

Only example viscous solution that is completely irrotational. $\sigma_{R\theta}$ exists due to net viscous force on fluid element.

$$\sigma_{R\theta} = \mu\left[R^{-1}\frac{\partial u_r}{\partial\theta} + R\frac{\partial}{\partial R}\left(\frac{u_\theta}{R}\right)\right] = -\frac{2\mu\Omega_1 R_1^2}{R^2} \qquad P = A \times \text{stress} \times V$$

Mechanical power per unit length $= (2\pi R_1)\sigma_{R\theta}u_\theta$, which equals the integrated dissipation.

Exercise 9.15. Consider a solid cylinder of radius a, steadily rotating at angular speed Ω in an infinite viscous fluid. The steady solution is irrotational: $u_\theta = \Omega a^2/R$. Show that the work done by the external agent in maintaining the flow (namely, the value of $2\pi R u_\theta \tau_{r\theta}$ at $R = a$) equals the viscous dissipation rate of fluid kinetic energy in the flow field.

Solution 9.15. Using the given velocity field, the shear stress is:

$$\tau_{R\varphi} = \mu R \frac{\partial}{\partial R}\left(\frac{u_\varphi}{R}\right) = \mu \Omega a^2 R \frac{\partial}{\partial R}\left(\frac{1}{R^2}\right) = -2\mu\Omega a^2 \frac{1}{R^2}.$$

The work done per unit height $= \left\{2\pi a \tau_{R\varphi} u_\varphi\right\}_{R=a} = 2\pi a \cdot 2\mu\Omega \cdot \Omega a = 4\pi\mu a^2\Omega^2$.

From (4.58) the viscous dissipation rate of kinetic energy per unit volume for an incompressible flow is $\rho\varepsilon = 2\mu S_{ij}S_{ij}$, where ε is the viscous dissipation of kinetic energy per unit mass. For the given flow field there is only one non-zero independent strain component:

$$S_{R\varphi} = S_{\varphi R} = \frac{R}{2}\frac{\partial}{\partial R}\left(\frac{u_\varphi}{R}\right) = \frac{\Omega a^2}{2} R \frac{\partial}{\partial R}\left(\frac{1}{R^2}\right) = -\Omega a^2 \frac{1}{R^2}.$$

Therefore:

$$\rho\varepsilon = 2\mu S_{ij}S_{ij} = 2\mu\left(S_{R\varphi}^2 + S_{\varphi R}^2\right) = 4\mu\Omega^2 \frac{a^4}{R^4},$$

so the kinetic energy dissipation rate per unit height is:

$$\int_a^\infty \rho\varepsilon 2\pi R dR = 8\pi\mu\Omega^2 a^4 \int_a^\infty \frac{1}{R^3} dR = 4\pi\mu\Omega^2 a^2,$$

which equals the work done turning the cylinder.

$$\frac{\partial}{\partial x_j}\left(U_i \sigma_{ij}\right) = \sigma_{ij}\frac{\partial U_i}{\partial x_j} + U_i \frac{\partial \sigma_{ij}}{\partial x_j}$$

Total work of surface force	Deformation work w/o $\underline{a}$ and lost to internal energy	Increase of KE since contributes fluid $\underline{a}$

$$\sigma_{ij}\frac{\partial U_i}{\partial x_j} = \sigma_{ij}\left(\varepsilon_{ij} + \omega_{ij}\right) = \sigma_{ij}\varepsilon_{ij}$$

$\sigma_{ij}\omega_{ij} = 0$ since it is the product of a symmetric and an anti-symmetric tensor.

$$\sigma_{ij}\frac{\partial U_i}{\partial x_j} = \left[-\left(p + \frac{2}{3}\mu\nabla\cdot\underline{U}\right)\delta_{ij} + 2\mu\varepsilon_{ij}\right]\varepsilon_{ij}$$

$$\sigma_{ij}\frac{\partial U_i}{\partial x_j} = -p\nabla\cdot\underline{U} + \underbrace{2\mu\varepsilon_{ij}\varepsilon_{ij} - \frac{2}{3}\mu\left(\nabla\cdot\underline{U}\right)^2}_{\boxed{\varphi}}$$

$\boxed{\text{Since } \varepsilon_{ij}\delta_{ij} = \varepsilon_{ii} = \nabla\cdot\underline{U}}$

$$\sigma_{ij}\frac{\partial U_i}{\partial x_j} = -p\nabla\cdot\underline{U} + \varphi$$

(2) $u_\theta(R) = \Omega_2 R$ solid body rotation

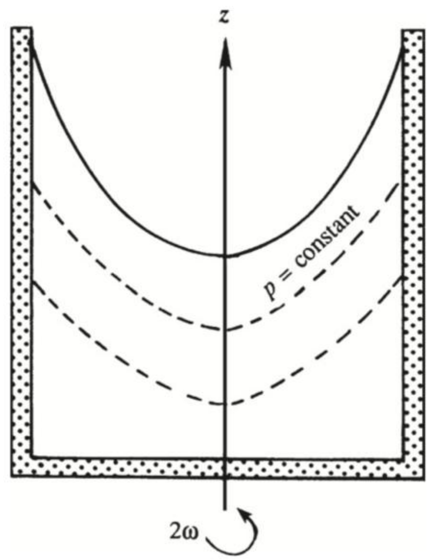


FIGURE 5.2 The steady flow field of a viscous liquid in a steadily rotating tank is solid body rotation. When the axis of rotation is parallel to the (downward) gravitational acceleration, surfaces of constant pressure in the liquid are paraboloids of revolution.