Kelvin–Helmholtz Inviscid Shear Layer Instability

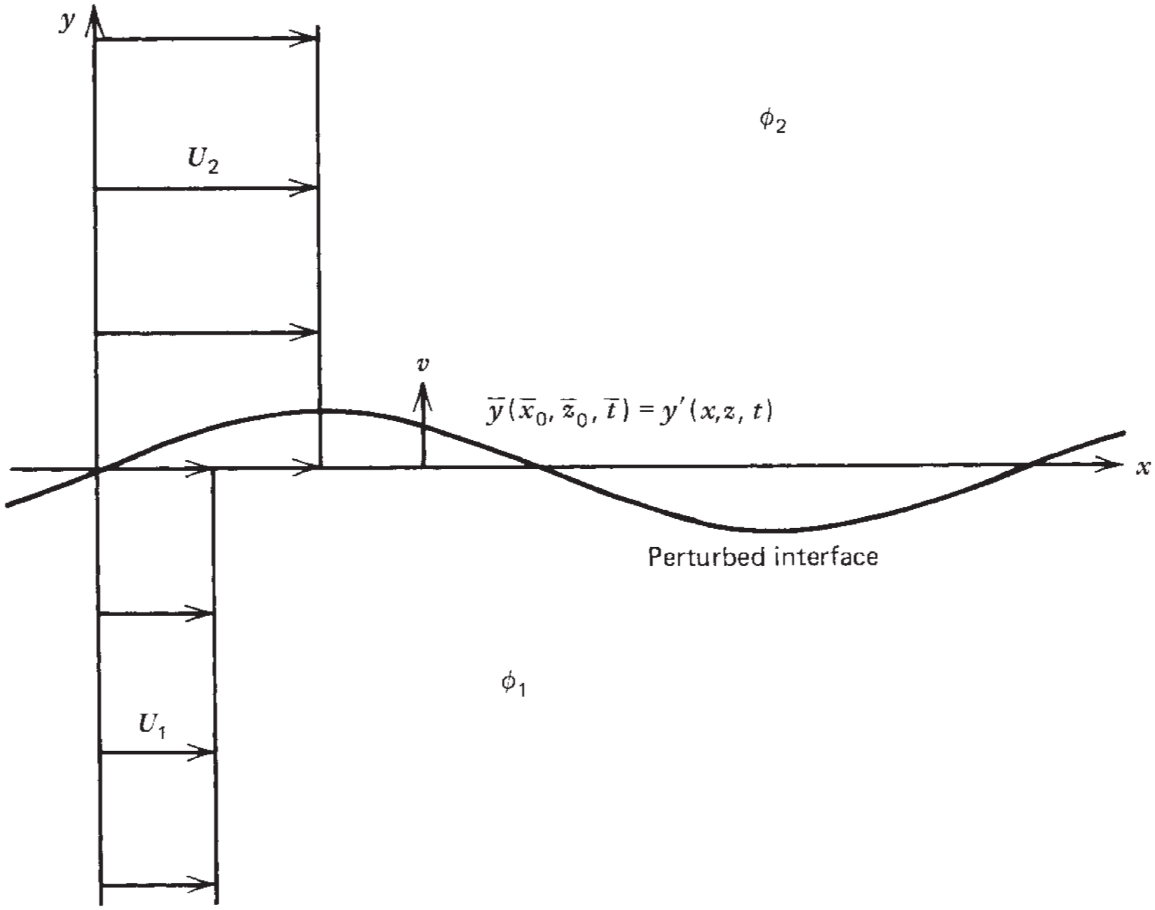

Figure 25.2 Shear layer nomenclature.

Realizable Splitter plate with ΔU uniform streams, wherein perturbation increases downstream, i.e., stability of inviscid vortex sheet. Recall previous discussion Chapter 3(5b) of viscous diffusion of vortex sheet:

$$u = U\,\mathrm{erf}(\eta)$$

$$\eta = \frac{y}{2\sqrt{\nu t}}, \quad \delta = \pm 5.52\sqrt{\nu t}$$

$$U_1 = -U \quad U_2 = U$$

$\nabla^2\phi_1 = 0 \quad \nabla^2\phi_2 = 0 \quad \nabla\phi_1 = U_1 \quad y \to -\infty \quad \nabla\phi_2 = U_2 \quad y \to \infty$

i.e., perturbations die out far from the interface

Kinematic free surface BC: $\frac{DF}{Dt} = 0$ on $y = y'$

$F = y - f(x, z, t) \quad y' = f(x, z, t)$ surface function

$$\frac{\partial F}{\partial t} + \underline{V} \cdot \nabla F = 0$$

$\underline{V_1} = \nabla\phi_1 = u_1\hat{\imath} + v_1\hat{\jmath} + w_1\hat{k}$

$\nabla F = -y'_x\hat{\imath} + \hat{\jmath} - y'_z\hat{k}$

$-y'_t - u_1 y'_x + v_1 - w_1 y'_z = 0$ or

$v_1 = y'_t + u_1 y'_x + w_1 y'_z = \frac{\partial\phi_1}{\partial y} \quad y = y'$

similarly for upper fluid

$v_2 = y'_t + u_2 y'_x + w_2 y'_z = \frac{\partial\phi_2}{\partial y} \quad y = y'$

Dynamic free surface BC: pressure continuous across the interface

$\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} = C(t)$ unsteady Bernoulli equation

$\phi_{1t} + \frac{1}{2}(\nabla\phi_1)^2 - C_1 = \phi_{2t} + \frac{1}{2}(\nabla\phi_2)^2 - C_2$ on $y = y'$

Basic flow satisfies same problem on $y' = 0$

$$C_1 - \frac{1}{2}U_1^2 = C_2 - \frac{1}{2}U_2^2$$

Next introduce perturbations from the basic flow along with $\pm\infty$ BC

$\phi_1 = U_1 x + \phi_1' \qquad \nabla^2\phi_1' = 0 \quad \nabla\phi_1' = 0 \quad y = -\infty$

$\phi_2 = U_2 x + \phi_2' \qquad \nabla^2\phi_2' = 0 \quad \nabla\phi_2' = 0 \quad y = \infty$

Kinematic BC is linearized and applied on $y = 0$

$v_1' = y_t' + U_1 y_x' = \phi_{1y}'$

$v_2' = y_t' + U_2 y_x' = \phi_{2y}'$

on $y = 0$

Similarly dynamic BC:

$\phi_{1t}' + U_1\phi_{1x}' = \phi_{2t}' + U_2\phi_{2x}'$

on $y = 0$

Assume normal mode perturbation solutions

$$\begin{Bmatrix} y' \\ \phi_1' \\ \phi_2' \end{Bmatrix} = \begin{Bmatrix} \hat{y} \\ \widehat{\phi_1}(y) \\ \widehat{\phi_2}(y) \end{Bmatrix} \exp[i(\alpha x + \beta z - \alpha c t)]$$

$c = c_R + i c_I =$ complex wave speed

$-i\alpha c = -\alpha(i c_R - c_I)$

$c_I > 0 \qquad$ unstable and grows exponentially in time

$\alpha =$ real $\quad \beta =$ real $\quad \omega = 2\pi f = \alpha c_R \quad \lambda = 2\pi/k$

$c_\phi =$ phase velocity $= \frac{\alpha c_r}{k} = \omega/k = \lambda f$ (for $\beta = 0, k = \alpha$ and $c_\phi = c_R$

$\hat{y} =$ constant $=$ original amplitude interface displacement (keys size perturbed quantities).

Substitution ϕ_1', ϕ_2' into Laplace with $\pm\infty$ BC

$$\widehat{\phi_1}(y) = A_1\, exp(ky) \qquad k = (\alpha^2 + \beta^2)^{1/2}$$

$$\widehat{\phi_2}(y) = A_2\, exp(-ky) \qquad \underline{k} = (\alpha, 0, \beta)$$

Using above and substitution y', ϕ_1', ϕ_2' in kinematic BC

$$A_1 = i\alpha \frac{\hat{y}}{k}(U_1 - c)$$

$$A_2 = -i\alpha \frac{\hat{y}}{k}(U_2 - c)$$

Then in dynamic BC

$$(U_1 - c)^2 = -(U_2 - c)^2$$

$$c = c_R + ic_I = \frac{1}{2}(U_1 + U_2) \pm i\frac{1}{2}|U_2 - U_1|$$

+ Sign unstable ALL wavenumbers α, β, i.e., all shear layers $U_1 \neq U_2$ inviscid unstable all wavelength disturbance!

Growth rate is $\exp(\alpha c_I t)$, i.e., for waves with $k = \alpha$ will grow fastest with phase speed

$$c_\phi = \frac{\alpha c_r}{k} = c_R = \frac{1}{2}(U_1 + U_2)$$

Disturbance travel at the average speed base flow

Viscous effects that spread "diffuse" the velocity profile stabilize the flow for λ ≈ shear layer thickness, whereas longer waves governed by inviscid analysis

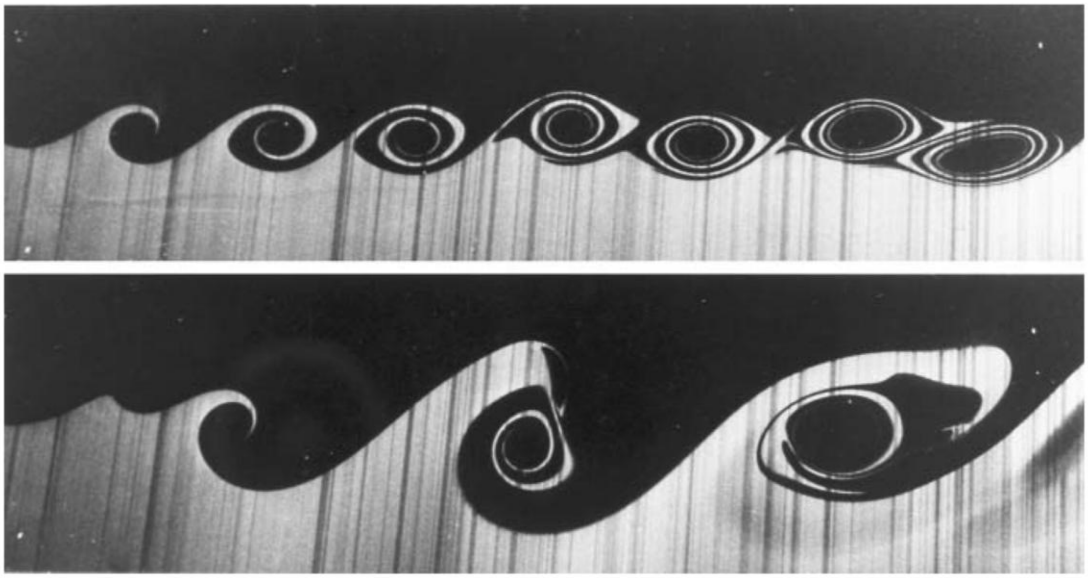

Figure 25.1 Kelvin–Helmholtz instability of a shear layer. The lower water stream, which contains a fluorescent dye, moves slower than the upper stream. A perturbation is introduced to initiate the growth in a regular pattern. Frequency is halved in the lower picture. Courtesy of F. A. Roberts, P. E. Dimotakis, and A. Roshko, California Institute of Technology.

$\nabla^2\phi_1' = 0 \qquad \phi_1' = \widehat{\phi_1}(y)e^{\gamma}, \quad \gamma = i(\alpha x + \beta z - \alpha c t)$

$\phi_{1xx}' + \phi_{1yy}' + \phi_{1zz}' = 0$

$-\alpha^2\widehat{\phi_1} - \beta^2\widehat{\phi_1} + \widehat{\phi_{1yy}} = 0 \Rightarrow \widehat{\phi_1}(y) = A_1 e^{ky} \qquad k = (\alpha^2 + \beta^2)^{1/2}$

$-\alpha^2 A_1 - \beta^2 A_1 + k^2 A_1 = 0$

$\phi_1' = A_1 e^{ky} e^{\gamma}$

$\phi_2' = A_2 e^{-ky} e^{\gamma}$

$\phi_1 = U_1 x + A_1 e^{ky} e^{\gamma} \qquad \phi_2 = U_2 x + A_2 e^{-ky} e^{\gamma} \qquad y' = \hat{y} e^{\gamma}$

Kinematic BC

$$\phi'_{1_y} = A_1 k e^{ky} e^{\gamma}$$

$$y'_x = \hat{y}\, e^{\gamma}(i\alpha) \qquad y'_t = \hat{y}\, e^{\gamma}(-i\alpha c)$$

$$A_1 k e^{ky} e^{\gamma} = \hat{y} e^{\gamma}(-i\alpha c) + U_1 \hat{y} e^{\gamma}(i\alpha)$$

$$y = 0 \quad A_1 k = \hat{y}(-i\alpha c) + U_1 \hat{y}(i\alpha)$$

$$A_1 = \frac{i\alpha\hat{y}(U_1 - c)}{k} \qquad A_2 = \frac{-i\alpha\hat{y}(U_2 - c)}{k}$$

Dynamic BC

$$A_1(-i\alpha c) + U_1 A_1(i\alpha) = A_2(-i\alpha c) + U_2 A_2(i\alpha)$$

$$A_1(i\alpha)(U_1 - c) = A_2(i\alpha)(U_2 - c)$$

$$(U_1 - c)^2 = -(U_2 - c)^2$$

$$U_1^2 - 2U_1 c + c^2 = -U_2^2 + 2U_2 c - c^2$$

$$2c^2 - 2(U_1 + U_2)c + U_1^2 + U_2^2 = 0$$

$$ax^2 + bx + c = 0, \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$c = \underbrace{\frac{1}{2}(U_1 + U_2)}_{c_r} \pm \frac{1}{4}[4(U_1 + U_2)^2 - 8(U_1^2 + U_2^2)]^{1/2}$$

$$4(U_1^2 + 2U_1U_2 + U_2^2) - 8U_1^2 - 8U_2^2$$

$$-4U_1^2 + 8U_1U_2 - 4U_2^2 = -4(U_1 - U_2)^2$$

$$\pm\frac{1}{4}\sqrt{-4(U_2 - U_1)^2} = \pm \underbrace{i\frac{1}{2}|U_2 - U_1|}_{c_i} \quad c = c_r + ic_i$$

+ Sign unstable −Sign stable decay disturbance