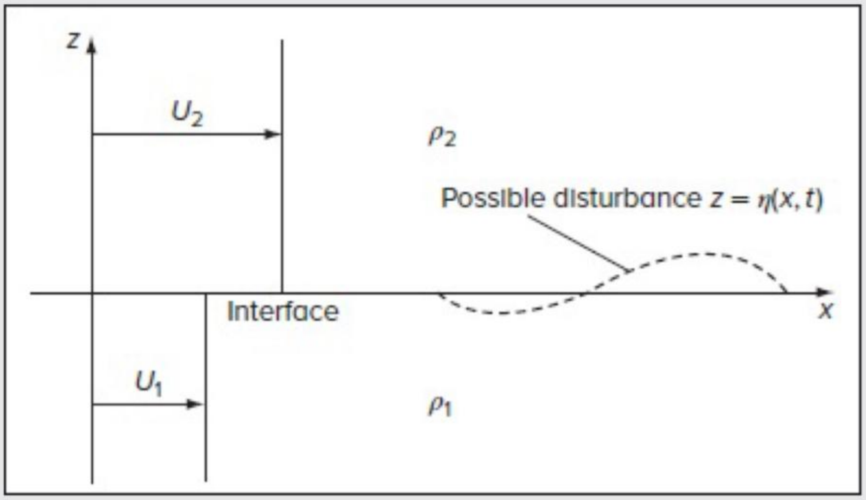
Wind generated waves: Kelvin-Helmholtz Instability



Horizontal interface divides two uniform flows of different U and ρ

ρ =constant, $\mu = 0$, $\underline{\omega} = 0$ in each flow

FIGURE 5-2

Sketch and nomenclature for the Kelvin-Helmholtz interfacial instability.

Step 1 Basic Flow

$z < 0$: $\quad \phi_1 = U_1 x \quad p_1 = p_0 - \rho_1 g z$

$z > 0$: $\quad \phi_2 = U_2 x \quad \mathrm{p}_2 = \mathrm{p}_0 - \rho_2 g z$

Both flows have velocity potential and hydrostatic $p(z)$

Since $\mu = 0$ tangential slip at interface: vortex sheet with discontinuity in velocity. Recall Chapter 3 (5b): potential flow vortex sheet and diffusion vortex sheet.

Step 2 Add disturbance

$$\phi_1 = U_1 x + \widehat{\phi_1}(x, z, t) \qquad \phi_2 = U_2 x + \widehat{\phi_2}(x, z, t)$$

Interface BC's:

Dynamic ⇒ pressure continuous across interface

unsteady Bernoulli $p_i = \rho_i C_i - \rho_i \frac{\partial \phi_i}{\partial t} - \frac{\rho_i}{2} |\nabla \phi_i|^2 - \rho_i g z$

For basic flow on $z = 0$ $\rho_1 C_1 - \frac{1}{2}\rho_1 U_1^2 = \rho_2 C_2 - \frac{1}{2}\rho_2 U_2^2$

For perturbed flow on $z = \eta$

$$\rho_1 C_1 - \rho_1 \frac{\partial \phi_1}{\partial t} - \frac{\rho_1}{2} |\nabla \phi_1|^2 - \rho_1 g z$$
$$= \rho_2 C_2 - \rho_2 \frac{\partial \phi_2}{\partial t} - \frac{\rho_2}{2} |\nabla \phi_2|^2 - \rho_2 g z$$

Kinematic ⇒ vertical velocities = interfacial motion or equivalently $F = z - \eta(x, t)$ is a material surface for which particles which lie on surface stay on surface, i.e., surface is stream surface.

On $z = \eta$ $\frac{DF}{Dt} = 0 = \frac{\partial F}{\partial t} + \underline{V_i} \cdot \nabla \eta$

$$\underline{V_1} = (U_1 + \widehat{\phi_{1x}})\hat{\imath} + \widehat{\phi_{1z}}\hat{k} \quad \underline{V_2} = (U_2 + \widehat{\phi_{2x}})\hat{\imath} + \widehat{\phi_{2z}}\hat{k}$$

$$\nabla F = F_x \hat{\imath} + F_z \hat{k} = -\eta_x \hat{\imath} + \hat{k}$$

$$-\eta_t - (U_1 + \widehat{\phi_{1x}})\eta_x + \widehat{\phi_{1z}} = 0 \qquad -\eta_t - (U_2 + \widehat{\phi_{2x}})\eta_x + \widehat{\phi_{2z}} = 0$$

Far field BC: disturbances → 0 for large z

$$\nabla \widehat{\phi_2} \to 0 \quad z \to \infty \qquad \nabla \widehat{\phi_1} \to 0 \quad z \to -\infty$$

Step 3 Subtract basic flow from disturbance equation

$$\nabla^2\phi_1 = \nabla^2\widehat{\phi_1} = 0 \qquad \nabla^2\phi_2 = \nabla^2\widehat{\phi_2} = 0$$

Both basic and disturbance equations satisfy Laplace equation

Step 4

Linearize equations

Disturbances ≪ basic flow; therefore, neglect products of disturbances and apply BC on $z = 0$

Dynamic BC

$$\rho_1 C_1 - \rho_1\widehat{\phi_{1t}} - \frac{\rho_1}{2}\left[\left(U_1 + \widehat{\phi_{1x}}\right)^2 + \widehat{\phi_{1z}^2}\right] - \rho_1 g\eta =$$

$$\rho_1 C_2 - \rho_2\widehat{\phi_{2t}} - \frac{\rho_2}{2}\left[\left(U_2 + \widehat{\phi_{2x}}\right)^2 + \widehat{\phi_{2z}^2}\right] - \rho_2 g\eta \qquad \text{on } z = 0$$

$$\rho_1 C_1 - \rho_1 C_2 = \frac{\rho_1}{2}{U_1}^2 - \frac{\rho_2}{2}{U_2}^2 \qquad \text{on } z = 0$$

$$\rho_1\left[\widehat{\phi_{1t}} + U_1\widehat{\phi_{1x}} + g\eta\right] = \rho_2\left[\widehat{\phi_{2t}} + U_2\widehat{\phi_{2x}} + g\eta\right] \text{ on } z = 0$$

Kinematic BC

$$\eta_t = -U_1\eta_x + \widehat{\phi_{1z}} = -U_2\eta_x + \widehat{\phi_{2z}} \qquad \text{on } z = 0$$

Step5 Assume normal mode disturbance (similar separation of variables)

$\widehat{\phi_J} = A_j(z)e^{i(\alpha x-\sigma t)}$

$\widehat{\phi_1} = A_1(z)e^{i(\alpha x-\sigma t)}$ $\quad \widehat{\phi_2} = A_2(z)e^{i(\alpha x-\sigma t)}$

α = wave number

σ = temporal growth rate: $\sigma_r + i\sigma_i$

$-i\sigma = -i\sigma_r + \sigma_i \quad \Rightarrow \quad \sigma_i > 0$ unstable

Step 6 Solve eigenvalue problem

$\nabla^2\widehat{\phi_1} = 0 = \widehat{\phi_{1,xx}} + \widehat{\phi_{1,zz}}$

$\widehat{\phi_{1x}} = A_1 i\alpha e^{i(\alpha x-\sigma t)}$

$\widehat{\phi_{1xx}} = -A_1\alpha^2 e^{i(\alpha x-\sigma t)}$

$\widehat{\phi_{1zz}} = A_{1_{zz}}e^{i(\alpha x-\sigma t)}$

$-\alpha^2 A_1 + A_{1_{zz}} = 0$

$-\alpha^2 A_2 + A_{2_{zz}} = 0$

Exponential solutions $A_j = A_j' e^{\pm\alpha z}$

where ± determined based far field BC.

$\widehat{\phi_1} = A_1' e^{[i(\alpha x-\sigma t)+\alpha z]}$ $\qquad \widehat{\phi_2} = A_2' e^{[i(\alpha x-\sigma t)-\alpha z]}$

Also assume $\eta = \eta_0 e^{i(\alpha x-\sigma t)}$

$\eta_x = \eta_0(i\alpha)e^{i(\alpha x-\sigma t)}$ $\qquad$ $\eta_t = \eta_0(-i\sigma)e^{i(\alpha x-\sigma t)}$

$\widehat{\phi_{1z}} = A_1'\alpha e^{i(\alpha x-\sigma t)}e^{\alpha z}$ $\qquad$ $\widehat{\phi_{1x}} = A_1'(i\alpha)e^{i(\alpha x-\sigma t)}e^{\alpha z}$

$\widehat{\phi_{2z}} = A_2'(-\alpha)e^{i(\alpha x-\sigma t)}e^{-\alpha z}$ $\qquad$ $\widehat{\phi_{2x}} = A_2'(i\alpha)e^{i(\alpha x-\sigma t)}e^{-\alpha z}$

$\widehat{\phi_{1t}} = A_1'(-i\sigma)e^{i(\alpha x-\sigma t)}e^{\alpha z}$ $\qquad$ $\widehat{\phi_{2t}} = A_2'(-i\sigma)e^{i(\alpha x-\sigma t)}e^{-\alpha z}$

Kinematic BC

$z = 0$:

$$-U_1\eta_0(i\alpha) + \alpha A_1' = \eta_0(-i\sigma) = -U_2\eta_0(i\alpha) - \alpha A_2' = \eta_t = \eta_0(-i\alpha)$$

$\alpha A_1' = i\eta_0(\alpha U_1 - \sigma)$ $\qquad$ $A_1' = i\eta_0\left(U_1 - \frac{\sigma}{\alpha}\right)$

$\alpha A_2' = -i\eta_0(\alpha U_2 - \sigma)$ $\qquad$ $A_2' = -i\eta_0\left(U_2 - \frac{\sigma}{\alpha}\right)$

Dynamic BC

$z = 0$:

$$\rho_1[A_1'(-i\sigma) + U_1A_1'(i\alpha) + g\eta_0] = \rho_2[A_2'(-i\sigma) + U_2A_2'(i\alpha) + g\eta_0]$$

$$\rho_1[A_1'(-i\sigma) - U_1\eta_0(\alpha U_1 - \sigma) + g\eta_0]$$
$$= \rho_2[A_2'(-i\sigma) + U_2\eta_0(\alpha U_2 - \sigma) + g\eta_0]$$

$$\rho_1\left[i\cancel{\eta_0}\left(U_1 - \frac{\sigma}{\alpha}\right)(-i\sigma) - U_1\cancel{\eta_0}(\alpha U_1 - \sigma) + g\eta_0\right] =$$

$$\rho_2\left[-i\cancel{\eta_0}\left(U_2 - \frac{\sigma}{\alpha}\right)(-i\sigma) + U_2\cancel{\eta_0}(\alpha U_2 - \sigma) + g\cancel{\eta_0}\right]$$

$$\rho_1\left[-\left(U_1 - \frac{\sigma}{\alpha}\right)\sigma - U_1(\alpha U_1 - \sigma) + g\right]$$
$$= \rho_2\left[-\left(U_2 - \frac{\sigma}{\alpha}\right)\sigma + U_2(\alpha U_2 - \sigma) + g\right]$$

$$\rho_1\left[\left(U_1-\frac{\sigma}{\alpha}\right)(\sigma-U_1\alpha)+g\right]=\rho_2\left[-\left(U_2-\frac{\sigma}{\alpha}\right)(\sigma-U_2\alpha)+g\right]$$

$$\rho_1[(\alpha U_1-\sigma)(\sigma-U_1\alpha)+g\alpha]=\rho_2[-(\alpha U_2-\sigma)(\sigma-U_2\alpha)+g\alpha]$$

$$(\alpha U_1-\sigma)(\sigma-U_1\alpha)=-(\sigma-\alpha U_1)^2=-\sigma^2+2\alpha U_1\sigma-\alpha^2U_1^2$$

$$-(\alpha U_2-\sigma)(\sigma-U_2\alpha)=(\sigma-\alpha U_2)^2=\sigma^2-2\alpha U_2\sigma+\alpha^2U_2^2$$

$$\rho_1[-(\sigma-\alpha U_1)^2+g\alpha]=\rho_2[(\sigma-\alpha U_2)^2+g\alpha]$$

$$-\rho_1\sigma^2+2\rho_1\alpha U_1\sigma-\rho_1\alpha^2U_1^2+\rho_1 g\alpha-\rho_2\sigma^2+2\rho_2\alpha U_2\sigma-\rho_2\alpha^2U_2^2 \\ -\rho_2 g\alpha=0$$

$$-\sigma^2(\rho_1+\rho_2)+2\alpha\sigma(\rho_1U_1+\rho_2U_2)-\alpha^2(\rho_1U_1^2+\rho_2U_2^2) \\ +g\alpha(\rho_1-\rho_2)=0$$

$$\sigma^2+\sigma\left[\frac{-2\alpha(\rho_1U_1+\rho_2U_2)}{\rho_1+\rho_2}\right]+\frac{\alpha^2}{\rho_1+\rho_2}\left(\rho_1U_1^2+\rho_2U_2^2-\frac{g}{\alpha}(\rho_1-\rho_2)\right) \\ =0$$

$$ax^2+bx+c=0,\quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$a=1,\quad b=-\frac{2\alpha(\rho_1U_1+\rho_2U_2)}{\rho_1+\rho_2}$$

$$c=\frac{\alpha^2}{\rho_1+\rho_2}[\rho_1U_1^2+\rho_2U_2^2]-\alpha g\frac{\rho_1-\rho_2}{\rho_1+\rho_2}$$

$$b^2=\frac{4\alpha^2(\rho_1U_1+\rho_2U_2)^2}{(\rho_1+\rho_2)^2}$$

$$-4ac = -\frac{4\alpha^2(\rho_1 U_1^2 + \rho_2 U_2^2)}{\rho_1 + \rho_2} + 4\alpha g\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$$

$$(\rho_1 + \rho_2)^{-2}[(\rho_1 U_1 + \rho_2 U_2)^2 - (\rho_1 U_1^2 + \rho_2 U_2^2)(\rho_1 + \rho_2)]$$

$$\cancel{\rho_1^2 U_1^2} + 2\rho_1\rho_2 U_1 U_2 + \cancel{\rho_2^2 U_2^2} - \cancel{\rho_1^2 U_1^2} - \rho_1\rho_2 U_2^2 - \rho_1\rho_2 U_1^2 - \cancel{\rho_2^2 U_2^2}$$

$$\rho_1\rho_2(2U_1U_2 - U_1^2 - U_2^2) = -\rho_1\rho_2(U_1 - U_2)^2$$

$$\sigma = \frac{\alpha(\rho_1 U_1 + \rho_2 U_2)}{\rho_1 + \rho_2} \pm \left[\underbrace{\frac{\alpha g(\rho_1 - \rho_2)}{\rho_1 + \rho_2}}_{(1)} - \underbrace{\frac{\alpha^2\rho_1\rho_2(U_1 - U_2)^2}{(\rho_1 + \rho_2)^2}}_{(2)}\right]^{1/2}$$

$$= \sigma_r + i\sigma_i$$

$\sigma_i > 0$ unstable $\sqrt{-p} = i\sqrt{p}$ $-p < 0$

For (2) < (1) $\sigma_i = 0$ ∴ neutral stability

For (1) < (2) σ_i and + root $\sigma_i > 0$

$$\frac{\alpha g(\rho_1 - \rho_2)}{\rho_1 + \rho_2} < \frac{\alpha^2\rho_1\rho_2(U_1 - U_2)^2}{(\rho_1 + \rho_2)^2}$$

or $g(\rho_1 - \rho_2)(\rho_1 + \rho_2) < \alpha\rho_1\rho_2(U_1 - U_2)^2$

or $g(\rho_1^2 - \rho_2^2) < \alpha\rho_1\rho_2(U_1 - U_2)^2$

∴ if ΔU large enough or $(\rho_1^2 - \rho_2^2)$ small enough

or $\alpha = 2\pi/\lambda$ large enough flow unstable

(i.e. λ small) ⇒ vortex sheets always unstable

Step 7 not needed an analytical solution provides stability condition

FIGURE 5-3

Kelvin-Helmholtz breaking waves outlined by a billow-cloud formation. [*John Davidson Photos.Alamy Stock Photo*].