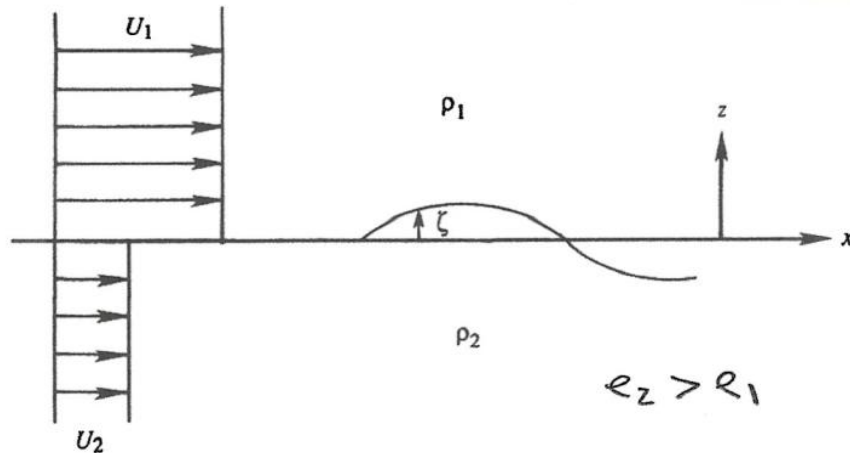


Kelvin-Helmholtz Instability

Enhanced momentum, heat, and moisture transport in the atmosphere plus routinely exploited in variety of geometries for the mixing two or more streams in engineering applications.



Two horizontal parallel streams different U . Infinite depths with zero thickness interface. Assume $\rho = \text{constant}$, $\underline{\omega} = 0$, and $\mu = 0$.

Step 1 Basic flow

$$\widetilde{\phi}_1 = U_1 x$$

$$\widetilde{\phi}_2 = U_2 x$$

Note tangential velocity and stresses are discontinuous across the interface: vortex sheet

Step 2 Add Disturbance

$$\widetilde{\phi}_1 = U_1 x + \phi_1 \quad \phi_1 \rightarrow 0 \quad z \rightarrow \infty$$

$$\widetilde{\phi}_2 = U_2 x + \phi_2 \quad \phi_2 \rightarrow 0 \quad z \rightarrow -\infty$$

$$\nabla^2 \phi_1 = 0 \quad \nabla^2 \phi_2 = 0 \quad z = \zeta(x, t)$$

Steps 3 and 4 Subtract Step 1 and linearize

Kinematic and dynamic interface conditions

Kinematic condition is that the vertical velocity is continuous across the interface and the interface moves with that velocity in the normal direction

$$F = z - \zeta \quad \frac{DF}{Dt} = 0 = \frac{\partial F}{\partial t} + \underline{V} \cdot \nabla F = -\zeta_t - (U_1 + u_1)\zeta_x - v_1\zeta_y + w_1 = 0$$

$$\nabla F = -\zeta_x \hat{i} - \zeta_y \hat{j} + \hat{k} \text{ or } \zeta_t + (U_1 + u_1)\zeta_x + v_1\zeta_y = w_1 = \phi_{1_z}$$

$$\nabla \phi_1 = u_1 \hat{i} + v_1 \hat{j} + w_1 \hat{k}$$

$$\frac{\partial \phi_1}{\partial z} = \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + (U_1 + u_1) \frac{\partial \zeta}{\partial x} + v_1 \frac{\partial \zeta}{\partial y} \quad \text{on } z = \zeta$$

$$\frac{\partial \phi_2}{\partial z} = \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + (U_2 + u_2) \frac{\partial \zeta}{\partial x} + v_2 \frac{\partial \zeta}{\partial y} \quad \text{on } z = \zeta$$

linearize:

$$\phi_{1_z} = \zeta_t + U_1 \zeta_x$$

$$\phi_{2_z} = \zeta_t + U_2 \zeta_x$$

$$\text{on } z = 0$$

Dynamic condition is that the stress must be continuous across the interface, i.e., $\widetilde{p}_1 = \widetilde{p}_2$ on $z = \zeta$ (neglecting surface tension).

$$\widetilde{p}_1 = P_1 + p_1 \quad \widetilde{p}_2 = P_2 + p_2$$

$$\frac{\partial \widetilde{\phi}_1}{\partial t} + \frac{1}{2} (\nabla \widetilde{\phi}_1)^2 + \frac{\widetilde{p}_1}{\rho} + gz = c_1 \quad \text{on } z = \zeta$$

$$\frac{\partial \widetilde{\phi}_2}{\partial t} + \frac{1}{2} (\nabla \widetilde{\phi}_2)^2 + \frac{\widetilde{p}_2}{\rho} + gz = c_2 \quad \text{on } z = \zeta$$

For basic flow $P_1 = P_2$ on $z = 0$

$$\rho_1 \left(\frac{1}{2} U_1^2 - c_1 \right) = \rho_2 \left(\frac{1}{2} U_2^2 - c_2 \right) \quad \text{on } z = 0$$

For the disturbed flow on $z = \zeta$

$$\begin{aligned} \rho_1 c_1 - \rho_1 \frac{\partial \widetilde{\phi}_1}{\partial t} - \frac{\rho_1}{2} [(U_1 + u_1)^2 + v_1^2 + w_1^2] - \rho_1 g \zeta \\ = \rho_2 c_2 - \rho_2 \frac{\partial \widetilde{\phi}_2}{\partial t} - \frac{\rho_2}{2} [(U_2 + u_2)^2 + v_2^2 + w_2^2] - \rho_2 g \zeta \end{aligned}$$

Subtracting disturbed flow from basic flow and linearize

$$\rho_1 [\phi_{1_t} + U_1 \phi_{1_x} + g \zeta] = \rho_2 [\phi_{2_t} + U_2 \phi_{2_x} + g \zeta] \quad \text{on } z = 0$$

ϕ_1 and ϕ_2 satisfy ∇^2 subject BC ∞ and linearized kinematic and dynamic interface conditions.

Step 5 Assume normal modes with $\underline{K} = (k, 0, 0)$

$$\phi_1(x, z, t) = A_1(z) e^{ik(x-ct)} \quad \phi_2(x, z, t) = A_2(z) e^{ik(x-ct)}$$

$$k = \text{real} \quad c = c_r + ic_i \quad -ikc = -ikc_r + kc_i$$

\therefore flow unstable $c_i > 0$. Note σt form more convenient convective cells.

$$\text{Step 6 Substitution in } \nabla^2: -k^2 A_1 + A_{1_{zz}} = 0 \quad -k^2 A_2 + A_{2_{zz}} = 0$$

Solutions $A_i e^{\pm kz}$ with \pm determined by BC ∞

$$\phi_1 = A_- e^{i(kx-ct)} e^{-kz} \quad \phi_2 = A_+ e^{i(kx-ct)} e^{kz}$$

$$\text{Similarly assume } \zeta = \zeta_0 e^{ik(x-ct)} \quad \delta = ik(x-ct)$$

$$\phi_{1_z} = A_- (-k) e^\delta e^{kz} \quad \zeta_t = \zeta_0 (-ikc) e^\delta$$

$$\phi_{2_z} = A_+ (k) e^\delta e^{-kz} \quad \zeta_x = \zeta_0 (ik) e^\delta$$

Kinematic BC

$$\phi_{1_z} - U_1 \zeta_x = \zeta_t = \phi_{2_z} - U_2 \zeta_x \quad \text{on } z = 0$$

$$A_- (-k) - U_1 \zeta_0 (ik) = \zeta_0 (-ikc) = A_+ (k) - U_2 \zeta_0 (ik)$$

$$kA_- = -\zeta_0 (ikU_1 - ikc)$$

$$kA_+ = \zeta_0 (ikU_2 - ikc)$$

Dynamic BC

$$\phi_{1_t} = A_- (-ikc) e^\delta e^{-kz} \quad \phi_{1_x} = A_- (ik) e^\delta e^{-kz}$$

$$\phi_{2_t} = A_+ (-ikc) e^\delta e^{kz} \quad \phi_{2_x} = A_+ (ik) e^\delta e^{kz}$$

$$\rho_1 [A_- (-ikc) + U_1 A_- (ik) + g\zeta_0] = \rho_2 [A_+ (-ikc) + U_2 A_+ (ik) + g\zeta_0]$$

on $z = 0$

$$\rho_1 [kA_- (-ic + iU_1) + g\zeta_0] = \rho_2 [kA_+ (-ic + iU_2) + g\zeta_0]$$

$$\rho_1 \left[\frac{kA_-}{\zeta_0} (-ikc + ikU_1) + kg \right] = \rho_2 \left[\frac{kA_+}{\zeta_0} (-ikc + ikU_2) + kg \right]$$

$$\rho_1 [-(-ikc + ikU_1)^2 + kg] = \rho_2 [(ikc + ikU_2)^2 + kg]$$

$$\begin{aligned}\rho_1[+(kc)^2 - 2k^2cU_1 + k^2U_1^2 + kg] \\ = \rho_2[-(kc)^2 + 2k^2cU_2 - k^2U_2^2 + kg]\end{aligned}$$

$$\rho_1 \left[c^2 - 2cU_1 + U_1^2 + \frac{g}{k} \right] = \rho_2 \left[-c^2 + 2cU_2 - U_2^2 + \frac{g}{k} \right]$$

$$c^2(\rho_1 + \rho_2) - c(2\rho_1U_1 + 2\rho_2U_2) + \rho_1U_1^2 + \rho_2U_2^2 + \frac{g}{k}(\rho_1 - \rho_2) = 0$$

$$c^2 + c \underbrace{\left[\frac{-2(\rho_1U_1 + \rho_2U_2)}{\rho_1 + \rho_2} \right]}_b + \underbrace{\left[\frac{\rho_1U_1^2 + \rho_2U_2^2}{\rho_1 + \rho_2} + \frac{g(\rho_1 - \rho_2)}{k(\rho_1 + \rho_2)} \right]}_c = 0$$

$$b^2 = \frac{4(\rho_1U_1 + \rho_2U_2)^2}{(\rho_1 + \rho_2)^2}$$

$$-4ac = -4(\rho_1U_1^2 + \rho_2U_2^2)(\rho_1 + \rho_2)^{-1} + \frac{4g(\rho_2 - \rho_1)}{k(\rho_1 + \rho_2)}$$

$$\begin{aligned}(\rho_1 + \rho_2)^{-2}[(\rho_1U_1 + \rho_2U_2)^2 - (\rho_1U_1^2 + \rho_2U_2^2)(\rho_1 + \rho_2)] \\ = -\frac{\rho_1\rho_2(U_2 - U_1)^2}{(\rho_1 + \rho_2)^2}\end{aligned}$$

$$c = \frac{\rho_2U_2 + \rho_1U_1}{\rho_2 + \rho_1} \pm \left[\underbrace{\frac{g(\rho_2 - \rho_1)}{k(\rho_2 + \rho_1)}}_{(1)} - \underbrace{\frac{\rho_1\rho_2(U_2 - U_1)^2}{(\rho_1 + \rho_2)^2}}_{(2)} \right]^{1/2}$$

for (2) < (1) c real for both solutions and solution stable

for (2) > (1) $c_i > 0$ and + solution unstable

$$\text{Step 7 } g(\rho_2^2 - \rho_1^2) < k\rho_1\rho_2(U_2 - U_1)^2$$

$$\text{using } (\rho_2 - \rho_1)(\rho_2 + \rho_1) = \rho_2^2 - \rho_1^2$$

ΔU high enough or $\Delta\rho$ small enough or k large enough

Note for each growing solution there is a corresponding decaying solution

Limiting cases:

$$(1) U_1 = U_2 = 0, \quad c = \pm \left[\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \right) \frac{g}{k} \right]^{1/2}$$

$\rho_2 < \rho_1$ unstable and $\rho_2 = \rho_1$ neutral stability and $ck =$ dispersion relation for waves on a density interface (Kundu et al. Section 8.7)

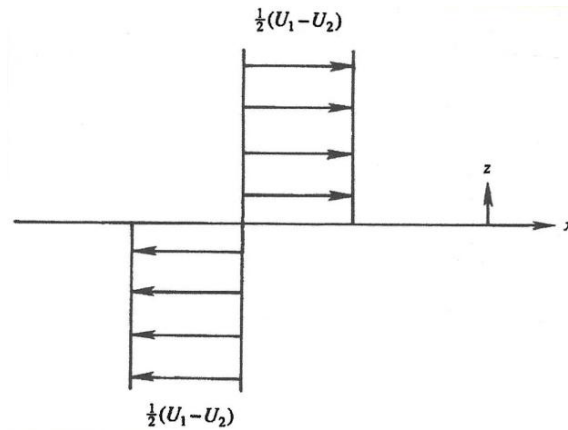
(2) For $U_1 \neq U_2$, k exists for which flow is unstable, i.e., flow is always unstable short waves $\lambda = \frac{2\pi}{k}$.

(3) For $\rho_1 = \rho_2$, ΔU discontinuous, i.e., vortex sheet with strength $\gamma = (U_2 - U_1)$

$$c = \frac{1}{2}(U_2 + U_1) \pm \frac{i}{2}(U_2 - U_1)$$

\therefore unstable all k and unstable waves move with phase velocity, i.e., average velocity = c_R

Basic flow in moving reference frame at average velocity c_R . A disturbance to the interface will be stationary in this reference frame.



Kelvin-Helmholtz caused by destabilizing effect shear overcoming stabilizing effects density stratification

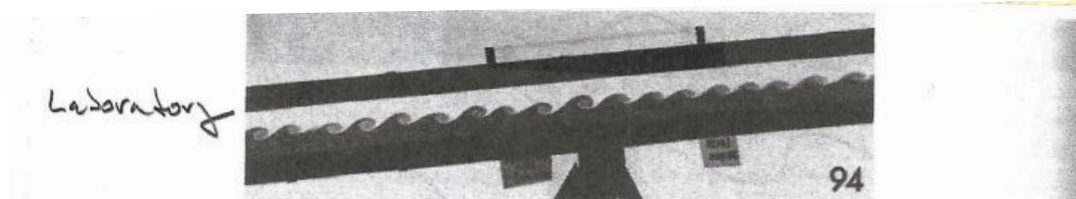


FIGURE 11.4 Kelvin-Helmholtz instability generated by tilting a horizontal channel containing two liquids of different densities. The lower layer is dyed and 18 wavelengths of the developing interfacial disturbance are shown. The mean flow in the lower layer is down the plane (to the left) and that in the upper layer is up the plane (to the right). S. A. Thorpe, *Journal of Fluid Mechanics*, 46, 299–319, 1971; reprinted with the permission of Cambridge University Press.

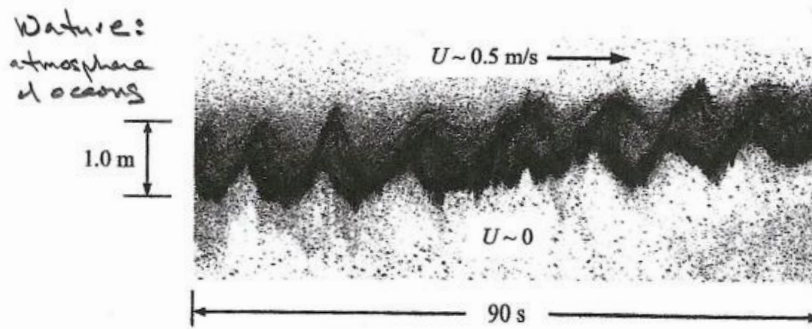


FIGURE 11.5 Natural overturning motions generated by the Kelvin-Helmholtz instability in a salt-stratified estuary at early ebb tide (see Lavery et al., 2013). The similarity of these structures to those shown in Figures 11.4 and 11.6 is striking. The image was created from a downward looking backscatter sonar system operating in the 375 kHz to 600 kHz band on a research vessel moving upstream into the river mouth. The upper fresh water layer from the Connecticut River is ~2 m deep. The lower saltwater layer is ~6 m deep. The Reynolds number of the flow based on the vertical extent of the overturning structures is ~500,000. The gradient Richardson number here is ~1/2 so the structures are not growing at this point (see Section 11.7, and Example 11.4). This image provided by, and used with the permission of, Jonathan Fincke & Andone C. Lavery of Woods Hole Oceanographic Institution.

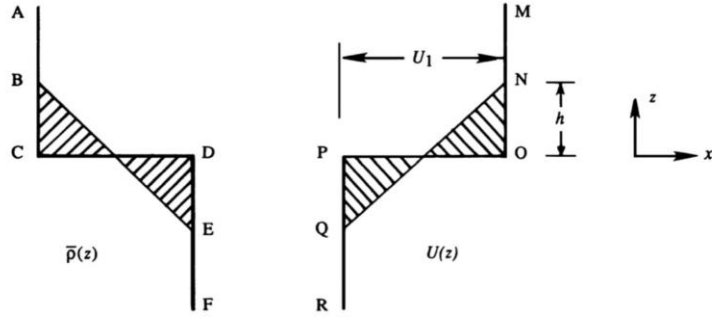


FIGURE 11.7 Smearing out of sharp density and velocity profiles, resulting in an increase of potential energy and a decrease of kinetic energy. When turbulent, the overturning eddies or billows shown in Figures 11.4 and 11.5 lead to cross-stream momentum transport and fluid mixing. The discontinuous profiles ACDF and MOPR evolve toward ABEF and MNQR as the instability develops.

Unstable waves at interface transform ACDF to ABEF for $\bar{\rho}(z)$ and MOPR to MNQR for $U(z)$. High density fluid depth DE raised upward (and mixed with lower density fluid depth BC), which means PE of system increased due instability. The required energy is drawn from the KE of the basic field, or the KE of the initial profile MOPR is $>$ than the final profile MNQR

For $U_2 = 0$, after mixing $U(z) = \frac{U_1}{2} \left(1 + \frac{z}{h}\right) \quad -h \leq z \leq h$

$$\bar{\rho}(z) = \rho_2 - \frac{(\rho_2 - \rho_1)}{2} \left(1 + \frac{z}{h}\right)$$

2D Δ KE in range $-h \leq z \leq h$ per unit width:

$$\begin{aligned} E_{initial} &= \frac{1}{2} \rho_1 U_1^2 h & E_{final} &= \frac{1}{2} \int_{-h}^h \bar{\rho}(z) U^2(z) dz \\ &= \frac{\rho_1 U_1^2 h}{3} + \frac{(\rho_2 - \rho_1) U_1^2 h}{12} \end{aligned}$$

For $(\rho_2 - \rho_1) \ll \rho_1$ KE decreases whereas total momentum $= \int \bar{\rho}(z) U(z) dz$ is nearly unchanged; since, $M_{initial} = \rho_1 U_1 h$ and $M_{final} = \frac{U_1 h (\rho_1 + 2\rho_2)}{3}$. If $\int \bar{\rho}(z) U(z) dz$ does not change, then $\int \bar{\rho}(z) U^2(z) dz$ decreases if gradients decrease.

Fig 11.4, 11.5 show advanced nonlinear stages of the instability, in which interface is a rolled-up layer of vorticity, which is replicated nonlinear numerical predictions

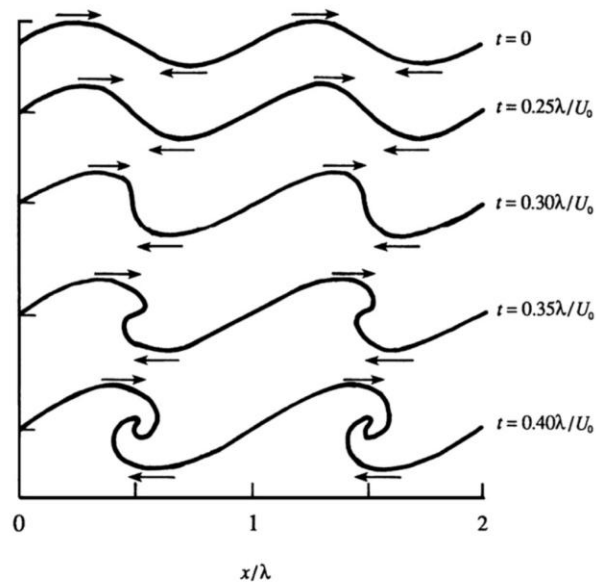


FIGURE 11.6 Nonlinear numerical calculation of the evolution of a vortex sheet that has been given a small transverse sinusoidal displacement with wavelength λ . The density difference across the interface is zero, and U_0 is the velocity difference across the vortex sheet. Here again, the similarity of the interface shape at the last time with the results shown in Figures 11.4 and 11.5 is striking. The smaller vertical displacements shown in Figures 11.4 and 11.5 are consistent with the effects of stratification that are absent from the calculations shown in this figure. *J. S. Turner, Buoyancy Effects in Fluids, 1973; reprinted with the permission of Cambridge University Press.*

Energy source of the K-H instability is the KE of the two streams.

The disturbances evolve to smear out gradients until they cannot grow further