Benard: Fluid Dynamics

$\nabla \times \underline{u} = \underline{\omega}$ (1) $\underline{u} = u_r \widehat{e_r} + u_\theta \widehat{e_\theta} = \nabla \times \underline{A}$ (2)

Where $\underline{A} = \left(0,0,\frac{\psi}{r\sin\theta}\right)$ = vector potential for axisymmetric flow in spherical coordinates (Appendix A) and used in Helmholtz decomposition, i.e., $\underline{V}^\omega = \nabla \times \underline{A}$. Note $\nabla \cdot \underline{A} = 0$.

$\nabla \times (\nabla \times \underline{A}) = \underline{\omega}$ (3)

$\nabla \times \left(\nabla \times \frac{\psi}{r\sin\theta}\widehat{e_\phi}\right) = \omega_\phi \widehat{e_\phi}$ (4)

$$-\frac{1}{r\sin\theta} L\psi\, \widehat{e_\phi} = \omega_\phi \widehat{e_\phi}$$

$L\psi = \psi_{rr} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\psi_\theta\right)$ (5)

using definition $\nabla \times \left(\nabla \times \frac{\psi}{r\sin\theta}\widehat{e_\phi}\right)$ or $\nabla^2\left(\frac{\psi}{r\sin\theta}\right)\widehat{e_\phi}$ in spherical coordinates. Appendix B.

Stokes Flow: $\nabla^2\underline{\omega} = 0$ $\quad\nabla \times \nabla \times \underline{\omega} = \nabla(\cancel{\nabla \cdot \underline{\omega}}) - \nabla^2\underline{\omega}$

$$\nabla \times (\nabla \times \underline{\omega}) = 0 = -\nabla^2\underline{\omega} \qquad \nabla \times (\nabla \times \omega_\phi \widehat{e_\phi}) = 0 = -\nabla^2\omega_\phi$$

$$\nabla \times \left(\nabla \times \left[\nabla \times \left(\nabla \times \frac{\psi}{r\sin\theta}\widehat{e_\phi}\right)\right]\right) = 0 = -\nabla^2\omega_\phi$$

$$\nabla \times \left(\nabla \times \left(-\frac{L\psi}{r\sin\theta}\widehat{e_\phi}\right)\right) = 0 = -\nabla^2\omega_\phi$$

(5) true any solution ψ including $L\psi$ $\therefore$

$$\nabla \times \left(\nabla \times \left(-\frac{L\psi}{r\sin\theta}\widehat{e_\theta}\right)\right) = -\frac{1}{r\sin\theta}L(L\psi)\widehat{e_\theta} = 0$$

i.e., $L(L\psi) = 0$ (6) Appendix B.

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0$$

BC: $(u_r, u_\theta) = (0,0) \quad r = a$ no slip

$u_r = \frac{\psi_\theta}{r^2 \sin\theta} \qquad u_\theta = -\frac{\psi_r}{r \sin\theta}$

i.e., $\psi_\theta(a,\theta) = 0$ and $\psi_r(a,\theta) = 0$

$\psi = \frac{1}{2} U r^2 \sin^2\theta$ uniform flow $r \to \infty$

Assume $\psi = f(r)\sin^2\theta$

$\psi_r = f' \sin^2\theta \qquad \psi_\theta = f 2 \sin\theta \cos\theta$

$\lim_{r\to\infty} f(r)/r^2 = U/2 \quad f(a) = 0 \quad f_r(a) = 0$

$L^2\psi = (f^{IV} - 4/r^2 f'' + 8/r^3 f' - 8/r^4 f)\sin^2\theta = 0$

General solution homogeneous equation is equidimensional equation solution r^β

$f(r) = c_1 r^4 + c_2 r^2 + c_3 r + c_4/r$

$c_1 = 0 \quad c_2 = U/2 \quad c_3 = -3Ua/4 \quad c_4 = Ua^3/4$

$f(r) = U(r^2/2 - 3ar/4 + a^3/4r)$

$$\psi(r,\theta) = \frac{1}{2} U r^2 \left(1 - \frac{3}{2}\left(\frac{a}{r}\right) + \frac{1}{2}\left(\frac{a}{r}\right)^3\right)\sin^2\theta$$

Comparison ψ inviscid and Stokes with same contour levels shows viscous streamlines spread further apart away from the sphere and shows presence of the sphere at further distances. Forward and aft symmetry due to vorticity flux from $r = a$ and outward viscous diffusion. Convection would cause asymmetry.

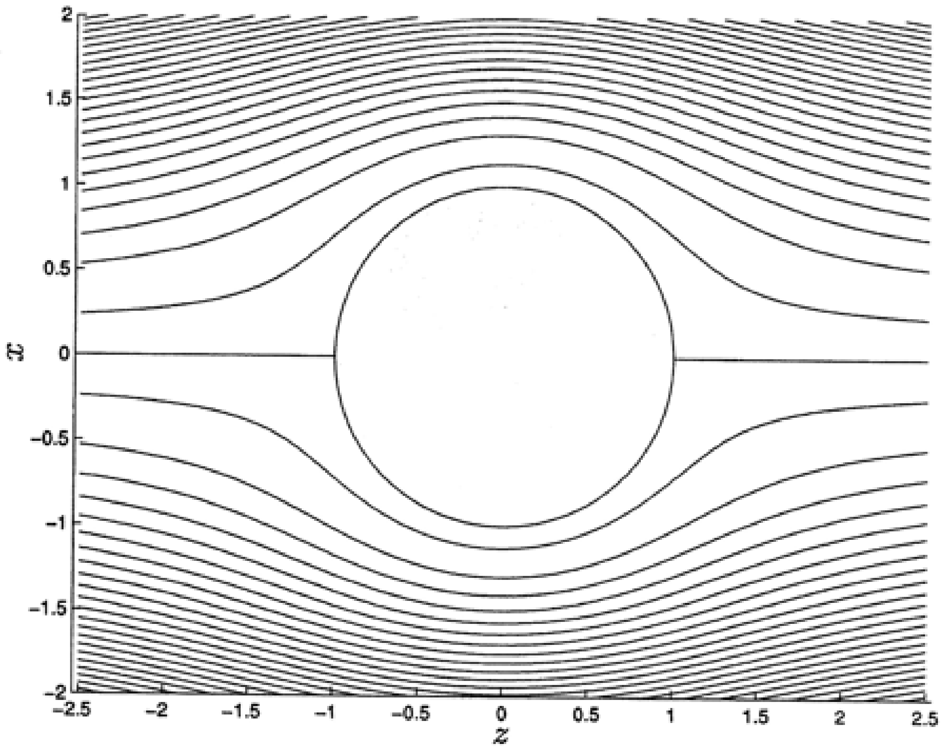

Figure 19.4. Streamlines of Stokes flow past a sphere.

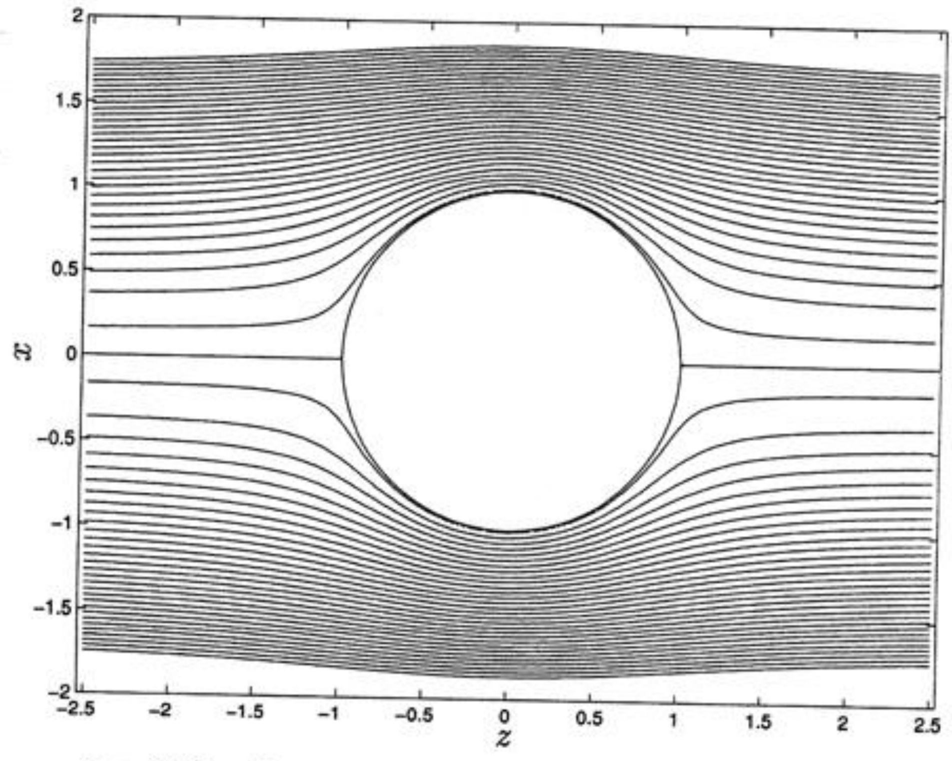

Figure 6.4. Streamlines corresponding to Eq. (6.11) in inviscid sphere flow.

Recall $\omega_\phi = -\frac{L\psi}{r\sin\theta} = -\frac{\sin\theta}{r}\left(f'' - \frac{2f}{r^2}\right)$

$f(r) = U(r^2/2 - 3ar/4 + a^3/4r)$

$f' = U(r - 3a/4 - a^3/4r^2)$

$f'' = U(1 + a^3/2r^3)$

$$U\left(1 + \frac{a^3}{2r^3}\right) - \frac{2U}{r^2}\left(\frac{r^2}{2} - \frac{3ar}{4} + \frac{a^3}{4r}\right)$$

$$\cancel{U} + \cancel{Ua^3/2r^3} - \cancel{U} + 3Ua/2r - \cancel{Ua^3/2r^3}$$

$$\omega_\phi = -\frac{\sin\theta}{r}\left(\frac{3Ua}{2r}\right) = -\frac{3}{2}\frac{Ua}{r^2}\sin\theta$$

Re-arrange $\psi(r,\theta) = \frac{1}{2}Ur^2\left(1 - \frac{3}{2}\left(\frac{a}{r}\right) + \frac{1}{2}\left(\frac{a}{r}\right)^3\right)\sin^2\theta$

$$\psi(r,\theta) = \underbrace{\frac{1}{2}Ur^2\sin^2\theta\left(1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right)}_{(1)} - \underbrace{\frac{3}{4}Uar\sin^2\theta}_{(2)}$$

(1) modifies ψ_I (2) responsible ω_ϕ

Compare potential flow: $\psi(r,\theta) = \frac{1}{2}Ur^2\sin^2\theta\left(1 - \left(\frac{a}{r}\right)^3\right)$

doublet + uniform flow

$u_r = U\cos\theta\left(1 - \left(\frac{a}{r}\right)^3\right)$ $\quad u_\theta = -U\sin\theta\left(1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right)$

$u_r(a,\theta) = 0$ $\quad u_\theta(a,\theta) = -\frac{3}{2}U\sin\theta$, i.e., only = 0 at 0, π

$$\psi_V - \psi_I = \frac{1}{2}Ur^2\sin^2\theta\left(\frac{3}{2}\left(\frac{a}{r}\right)^3\right) - \frac{3}{4}Uar\sin^2\theta$$

Forces on sphere

$$\nabla p = \mu\nabla^2\underline{u} == -\mu\nabla\times\underline{\omega}$$

$$\nabla\times\nabla\times\underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla^2\underline{u}$$

$$\nabla p = \mathrm{p}_r\,\widehat{e_r} + r^{-1}\mathrm{p}_\theta\,\widehat{e_\theta}$$

$$\underline{\omega} = \omega_\phi\widehat{e_\phi} \qquad \omega_\phi = -\frac{3}{2}\frac{Ua}{r^2}\sin\theta$$

$$\nabla = \frac{\partial}{\partial r}\widehat{e_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\widehat{e_\theta}$$

$$\nabla\times(\omega_\phi\widehat{e_\phi}) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\omega_\phi\sin\theta)\widehat{e_r} - \frac{1}{r}\frac{\partial}{\partial r}(r\omega_\phi)\widehat{e_\theta}$$

$$\frac{\partial\omega_\phi}{\partial\theta} = -\frac{3}{2}\frac{Ua}{r^2}\cos\theta$$

$$\frac{\partial\omega_\phi}{\partial r} = -\frac{3}{2}Ua\sin\theta\,(-2r^{-3}) = \frac{3Ua}{r^3}\sin\theta$$

$$r\omega_\phi = -\frac{3}{2}\frac{Ua}{r}\sin\theta$$

$$(r\omega_\phi)_r = \frac{3}{2}\frac{Ua}{r^2}\sin\theta$$

$$-\frac{1}{r}(r\omega_\phi)_r = -\frac{3}{2}\frac{Ua}{r^3}\sin\theta$$

$$\frac{1}{r\sin\theta}\left[\cos\theta\ \omega_\phi + \sin\theta\ \frac{\partial\omega_\phi}{\partial\theta}\right]$$

$$= \frac{1}{r\cancel{\sin\theta}}\left[-\frac{3}{2}\frac{Ua}{r^2}\cos\theta\cancel{\sin\theta} + \cancel{\sin\theta}\left(-\frac{3}{2}\frac{Ua}{r^2}\cos\theta\right)\right] = -\frac{3Ua}{r^3}\cos\theta$$

i.e., $p_r = \frac{3\mu Ua}{r^3}\cos\theta \qquad p_\theta = \frac{3\mu Ua}{2r^3}\sin\theta$

$$p - p_\infty = \int_0^r p_r \, dr + \int_0^\theta p_\theta \, r \, d\theta = -\frac{3}{2}\mu U a \frac{\cos\theta}{r^2}$$

$$3\mu U a \cos\theta \int_0^r r^{-3} \, dr = 3\mu U a \cos\theta \left(-\frac{1}{2r^2}\right)$$

$$= -\frac{3}{2}\mu U a \frac{\cos\theta}{r^2} + \cancel{f(\theta)}$$

$$\frac{3\mu U a}{2r^3}\int_0^\theta \sin\theta \, r \, d\theta = -\frac{3}{2}\mu U a \frac{\cos\theta}{r^2} + \cancel{f(r)}$$

$$p(r,\theta) = p_\infty - \frac{3}{2}\mu U a \frac{\cos\theta}{r^2}$$

axisymmetric $\therefore$ unlike potential flow there would be drag

$$C_p = \frac{\Delta p}{\rho U^2/Re} = -\frac{3\cos\theta}{(r/a)^2} \qquad Re = \frac{2aU}{\nu} \qquad \frac{\rho U^2}{2a\,U/\nu} = \frac{\mu U}{2a}$$

i.e., p scales like $\rho U^2/Re$ (low *Re*) vs. ρU^2 (high *Re*)

$F_i = \sigma_{ij} n_j = [-p\delta_{ij} + \tau_{ij}]n_j = -p n_i + \tau_{ij} n_j$ = force per unit area normal surface

component in $\hat{\imath}$ direction $= [\underbrace{-p\cos\theta + \tau_{rr}\cos\theta - \tau_{r\theta}\sin\theta}_{on\ r=a}]$

$$\tau_{rr} = 2\mu\frac{\partial u_r}{\partial r} = 2\mu U\cos\theta\left[\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right]$$

$$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right] = -\frac{3\mu U a^3}{2r^4}\sin\theta$$

$$\underbrace{\frac{3\mu U}{2a}\cos^2\theta}_{1/3} + 0 + \underbrace{\frac{3\mu U}{2a}\sin^2\theta}_{2/3} = \frac{3\mu U}{2a}$$

$$D = \frac{3\mu U}{2a} \times \underbrace{4\pi a^2}_{surface\ area} = 6\pi\mu a U$$

Re < 0.5 excellent, 10% error Re = 1

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 \underbrace{\pi a^2}_{projected\ area}} = 24/Re$$ Stokes drag law

$D/\mu a U = 6\pi$ proper form Stokes flow

Self-consistency

$$u_r = U\left(1 - \frac{3a}{2r} + \frac{1}{2}\left(\frac{a}{r}\right)^3\right)\cos\theta$$

$$u_\theta = -U\left(1 - \frac{3a}{4r} - \frac{1}{4}\left(\frac{a}{r}\right)^3\right)\sin\theta$$

which neglects convection. However, ratio $\underline{u} \cdot \nabla \underline{u} \,/\, \nu\nabla^2 \underline{u}$ only small close to sphere. $\nu\nabla^2\omega_\phi \downarrow$ and $r \uparrow$ such that at about 3D convection/diffusion ~ 1.

Correction O order solution: Oseen equations.

$\underline{U} = U\hat{\imath}$ $\nabla \cdot \underline{u} = 0$ $\rho\, \underline{U} \cdot \nabla \underline{u} = -\nabla p + \mu\nabla^2 \underline{u}$ still linear

Same Stokes near sphere and improvement far field.

Vorticity equation becomes

$$\underline{U} \cdot \nabla \underline{\omega} = \nu\nabla^2 \underline{\omega}$$

vorticity diffusion and convection downstream velocity $U\hat{\imath}$

$$C_D = \frac{24}{Re}\left(1 + \frac{3}{16}Re\right)$$

which is only small improvement for increased Re.