Stokes sphere solution using separation of variables and spherical coordinates, as per Appendix 9.2-A

$\nabla p = \mu \nabla^2 \underline{u}$

$0 = \nabla^2 \underline{\omega}$ note: $-\nabla \times \nabla \times \underline{\omega} = -\nabla(\nabla \cdot \underline{\omega}) + \nabla^2 \underline{\omega} = \nabla^2 \underline{\omega}$

only component of vorticity $\nabla \times \underline{u}$ is

$$\omega_\phi = \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)$$

$$\underline{u} = -\nabla \phi \times \nabla \psi = -\frac{\widehat{\iota_\phi}}{r \sin\theta} \times \left(\widehat{\iota_r} \psi_r + \widehat{\iota_\theta} \frac{\psi_\theta}{r} + \widehat{\iota_\phi} \frac{\psi_\phi}{r \sin\theta}\right)$$

$$= -\frac{\widehat{\iota_\theta}}{r \sin\theta} \psi_r + \frac{\widehat{z_r}}{\sin\theta} \frac{\psi_\theta}{r}$$

$$= \underbrace{\frac{\psi_\theta}{r \sin\theta}}_{u_r} \widehat{\iota_r} \; \underbrace{-\frac{\psi_r}{r \sin\theta}}_{u_\theta} \widehat{\iota_\theta}$$

$$\omega_\phi = -\frac{1}{r}\left[\frac{\psi_{rr}}{\sin\theta} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\psi_\theta\right)\right]$$

Combining with $\nabla^2 \omega_\phi = 0 \Rightarrow \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0 \quad (1)$

See Appendix 9.3-B

$$\left.\begin{array}{ll} \psi_\theta(a,\theta) = 0 & u_r = 0 \\ \psi_r(a,\theta) = 0 & u_\theta = 0 \end{array}\right\} r = a \quad \text{no slip}$$

$\psi(\infty, \theta) = \frac{1}{2} U r^2 \sin^2\theta$ uniform flow at ∞

Assume $\psi = f(r) \sin^2\theta$

i.e. separation of variables based far field solution

Substitution in (1)

$$f^{IV} - \frac{4f''}{r^2} + \frac{8f'}{r^3} - \frac{8f}{r^4} = 0$$

Equidimensional equation with solutions power series r^n and in this case n = -1,1,2,4

$$f = Ar^4 + Br^2 + Cr + \frac{D}{r}$$

∞ condition : $A = 0$ and $B = U/2$

$r = a$ condition : $C = -\frac{3Ua}{4}$ and $D = Ua^3/4$

$$\psi = Ur^2 \sin^2\theta \left[\frac{1}{2} - \frac{3a}{4r} + \frac{a^3}{4r^3}\right]$$

$$= \frac{U \sin^2\theta}{4}\left[\underbrace{2r^2}_{uniform\ flow} - \underbrace{3ar}_{Stokeslet} + \underbrace{\frac{a^3}{r}}_{dipole}\right]$$

$$u_r = U\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)$$

$$u_\theta = -U\sin\theta\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)$$

$$\nabla p = \mu\nabla^2\underline{V}$$

$$p = -\frac{3a\mu U\cos\theta}{2r^2} + p_\infty$$

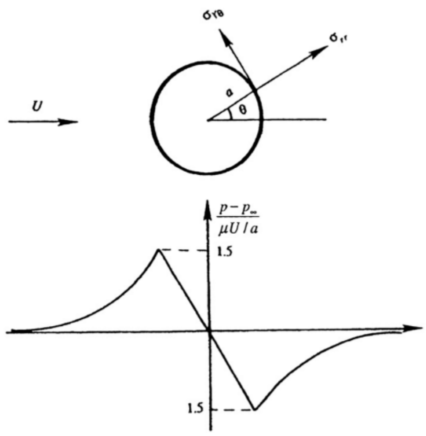


FIGURE 8.17 Creeping flow over a sphere. The upper panel shows the viscous stress components at the surface. The lower panel shows the pressure distribution in an axial (φ = const.) plane.

P_{max}/P_{min} at fwd/aft Stagnation points (π/0) $P_{max}/P_{min} = \pm\frac{3\mu U}{2a}$

$p \propto \mu$ antisymmetric + front − back create pressure drag

Viscous shear stress

$$\tau_{rr} = 2\mu \frac{\partial u_r}{\partial r} = 2\mu U \cos\theta \left[\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right]$$

$$\tau_{r\theta} = \mu \left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \underbrace{\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}}_{r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)}\right] = \frac{-\mu U \sin\theta}{r}\left(\frac{3a^3}{2r^3}\right)$$

$$F = -\int_0^{2\pi} \tau_{r\theta}\,|_{r=a} \sin\theta \; dA - \int_0^{2\pi} p|_{r=a} \cos\theta \; dA$$

$dA = 2\pi a^2 \sin\theta \; d\theta$ Surface area: $4\pi a^2$

Volume: $\frac{4}{3}\pi a^3$

$$F = \underbrace{4\pi\mu U a}_{2/3\ viscous} + \underbrace{2\pi\mu U a}_{1/3\ pressure} = 6\pi\mu U a$$

$\propto$ U

$\propto$ μ

$Re \ll 1$, but agrees EFD up to $Re = 1$

$\frac{F}{\mu U a} = 6\pi =$ constant since $\underline{V} \cdot \nabla \underline{V} = 0$, ρ not important

$$C_D = \frac{2F}{\rho U^2 \pi a^2} = \frac{24}{Re} \qquad Re = \frac{2a\rho U}{\mu}$$

$Re > 20$ separation, pressure drag increases

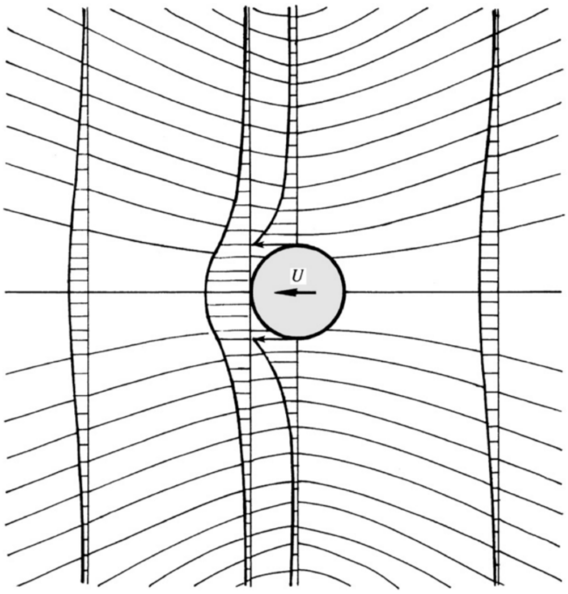


FIGURE 8.19 Streamlines and velocity distributions in Stokes' solution of creeping flow due to a moving sphere. Note the upstream and downstream symmetry, which is a result of complete neglect of nonlinearity.

$\psi - \text{uniform stream} = Ur^2 \sin^2\theta \left(-\frac{3a}{4r} + \frac{a^3}{4r^3}\right)$ Inertial refernce frame

Symmetric fore and aft due $\underline{V} \cdot \nabla \underline{V} = 0$ and no wake, i.e., change in direction. Just sign $-\underline{V}$ and $-p$

Nonuniformity Stokes solution: Oseen Approximation

Viscous force/$\forall =$ stress gradient $\sim \frac{\mu U a}{r^3} \quad r \rightarrow \infty$

Inertial force /$\forall$ = $\sim \rho u_r \frac{\partial u_\theta}{\partial r} \sim \frac{\rho U^2 a}{r^2} \quad r \rightarrow \infty$

$$\therefore \frac{inertial\ force}{viscous\ force} \sim \frac{\rho U a}{\mu} \frac{r}{a} = Re\ \frac{r}{a} \quad r \rightarrow \infty$$

Inertia not negligible for $\frac{r}{a} \sim \frac{1}{Re}$ no matter how small $Re,$ which occurs at distances of order $\nu/U.$

It can be shown that including 1st order term $\underline{V}$ is infinite at large distances, which is called Whitehead paradox, as was the case for the 0th order 2D solution, which was called Stokes paradox: singular perturbation problems

Oseen improvement:

$u = U + u', \quad v = v', \quad w = w'$

u', v', w' = Cartesian components perturbation $\underline{V} \ll U\, at\, r \to \infty$

e.g. x-momentum

$u\, u_x + v\, u_y + w\, u_z = U\, u'_x + \left(u' u'_x + v' u'_y + w' u'_z\right)$

Oseen equations

$$\rho U \frac{\partial u'_i}{\partial x} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u'_i$$

$$u'_i = (u', v', w') \quad \text{i.e.} \quad \underline{V} \cdot \nabla \underline{V} \sim U \frac{\partial \underline{u}}{\partial x}$$

Same order Stokes near body; however, in far field provide better approximation where solution ~ U

$$u' = v' = w' = 0 \quad r \to \infty$$

$$u' = -U, \quad v' = w' = 0 \quad r = 0$$

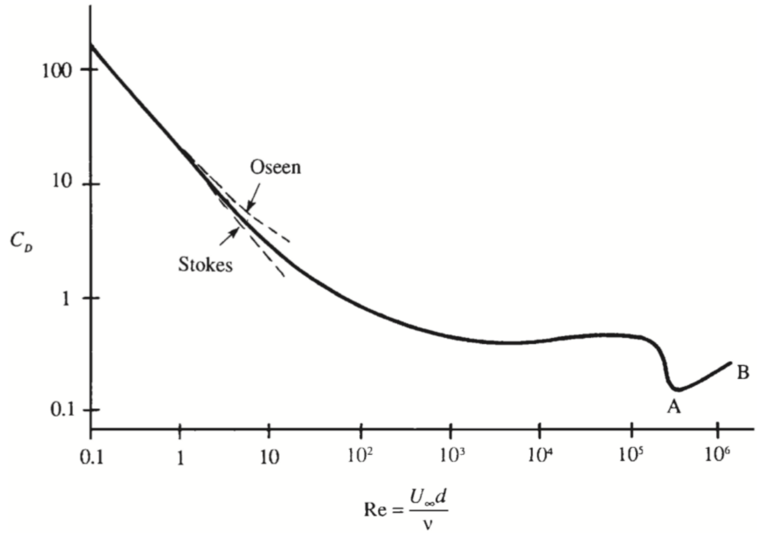
$$\frac{\psi}{Ua^2} = \left[\frac{r^2}{2a^2} + \frac{a}{4r}\right] \sin^2\theta - \frac{3}{Re}(1 + \cos\theta)\left\{1 - \exp\left[-\frac{Re\, r}{4a}(1 - \cos\theta)\right]\right\}$$

$$Re = 2aU/\nu \quad r/a \sim 1 \text{ recover Stokes solution}$$

$$C_D = \frac{24}{Re}\left(1 + \frac{3}{16}Re\right)$$

Lowest order solution uniformly valid near and far field

FIGURE 9.22 Measured drag coefficient, C_D, of a smooth sphere vs. $Re = U_\infty d/\nu$. The Stokes solution is $C_D = 24/Re$, and the Oseen solution is $C_D = (24/Re)\ (1 + 3Re/16)$; these two solutions are discussed at the end of Chapter 8. The increase of drag coefficient in the range A−B has relevance in explaining why the flight paths of sports balls bend in the air.



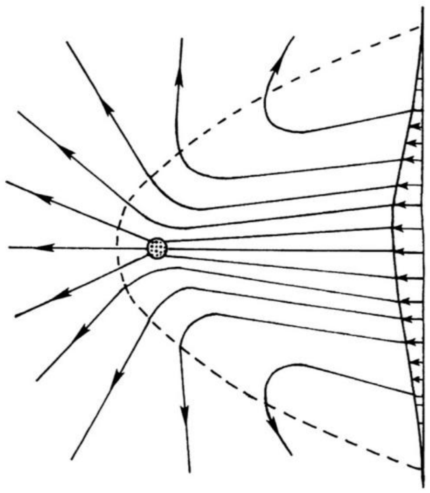


FIGURE 8.20 Streamlines and velocity distribution in Oseen's solution of creeping flow due to a moving sphere. Note the upstream and downstream asymmetry, which is a result of partial accounting for advection in the far field.

Oseen solution inertial reference frame. Flow no longer symmetric but has a wake where streamlines closer together and $\underline{V}$ larger in wake than front, whereas in Stokes expansion flow symmetric in wake than in front, whereas in moving refence frame flow slower in wake than in front. Advanced methods use matched asymptotic expansion techniques.

Exercise 9.45. Using the velocity field (8.49), determine the drag on Stokes' sphere from the surface pressure and the viscous surface stresses σ_{rr} and $\sigma_{r\theta}$.

Solution 9.45. There are pressure and shear stress contributions to the drag on a moving sphere at low Reynolds number. The pressure distribution is given by (8.50):

$$p(r,\theta) - p_\infty = -\frac{3\mu a U}{2r^2}\cos\theta .$$

The pressure drag can be obtained by integrating this result:

$$F_{pressure} = -\int_{surface} p(r=a,\theta)\mathbf{e}_r \cdot \mathbf{e}_z dS = -2\pi a^2 \int_{\theta=0}^{\theta=\pi} \mu\left(\frac{3U}{2a}\right)\cos^2\theta\sin\theta d\theta = 2\pi\mu U a$$

The viscous drag can be obtained from surface integrals of the viscous stresses:

$$F_{vicous} = -\int_{surface} \sigma_{r\theta}(r=a,\theta)\mathbf{e}_\theta \cdot \mathbf{e}_z dS + \int_{surface} \sigma_{rr}(r=a,\theta)\mathbf{e}_r \cdot \mathbf{e}_z dS$$

$$= -2\pi a^2 \int_{\theta=0}^{\theta=\pi} \sigma_{r\theta}(r=a,\theta)\sin^2\theta d\theta + 2\pi a^2 \int_{\theta=0}^{\theta=\pi} \sigma_{rr}(r=a,\theta)\cos\theta\sin\theta d\theta,$$

where $\sigma_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = -\frac{\mu U\sin\theta}{r}\left(\frac{3a^3}{2r^3}\right)$, and $\sigma_{rr} = 2\mu\frac{\partial u_r}{\partial r} = 2\mu U\cos\theta\left(\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right)$.

Thus, at $r = a$, $\sigma_{r\theta} \neq 0$, but $\sigma_{rr} = 0$, so

$$F_{vicous} = 3\pi\mu U a \int_{\theta=0}^{\theta=\pi} \sin^3\theta d\theta = 4\pi\mu U a .$$

Thus, one third of the drag comes from pressure forces and two thirds come from the shear stress. The total drag is the sum of these two contributions:

$$F_{drag} = F_{pressure} + F_{vicous} = 2\pi\mu U a + 4\pi\mu U a = 6\pi\mu U a .$$

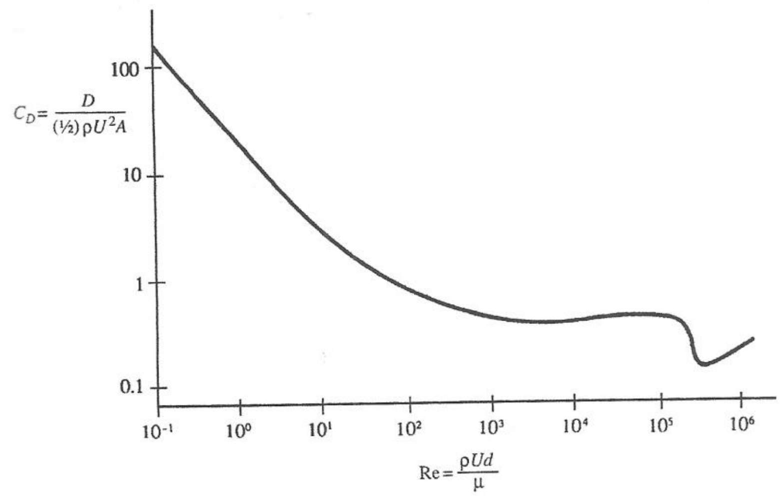


FIGURE 4.23 Coefficient of drag C_D for a sphere vs. the Reynolds number Re based on sphere diameter. At low Reynolds number $C_D \sim 1/Re$, and above $Re \sim 10^3$, $C_D \sim$ constant (except for the dip between $Re = 10^5$ and 10^6). These behaviors (except for the dip) can be explained by simple dimensional reasoning. The reason for the dip is the transition of the laminar boundary layer to a turbulent one, as explained in Chapter 10.

Exercise 4.75. From Figure 4.23, it can be seen that $C_D \propto 1/\mathrm{Re}$ at small Reynolds numbers and that C_D is approximately constant at large Reynolds numbers. Redo the dimensional analysis leading to (4.99) to verify these observations when:
a) Re is low and fluid inertia is unimportant so ρ is no longer a parameter.
b) Re is high and the drag force is dominated by fore-aft pressure differences on the sphere and μ is no longer a parameter.

Solution 4.75. Overall there are 5 parameters D = drag force, U, d, ρ, and μ.
a) When ρ is not a parameter, the dimensional analysis proceeds a follows. The parameter & units matrix is:

	D	U	d	μ
M	1	0	0	1
L	1	1	1	-1
T	-2	-1	0	-1

This rank of this matrix is three. There are 4 parameters and 3 independent units, so there will be 1 dimensionless group:

$$\Pi_1 = \frac{D}{\mu U d} = \text{const., or } \quad D = (const.)\mu U d.$$

Divide both sides of the last equation by $\frac{1}{2}\rho U^2 d^2 \pi/4$ to find:

$$\frac{D}{\frac{1}{2}\rho U^2 d^2 \pi/4} = C_D = \frac{(const.)\mu U d}{\frac{1}{2}\rho U^2 d^2 \pi/4} = const.\frac{\mu}{\rho U d} = \frac{const.}{\mathrm{Re}},$$

where the 2, 4, and π have been absorbed into the undetermined constant.
b) When μ is not a parameter, the dimensional analysis proceeds a follows. The parameter & units matrix is:

	D	U	d	ρ
M	1	0	0	1
L	1	1	1	-3
T	-2	-1	0	0

This rank of this matrix is three. There are 4 parameters and 3 independent units, so there will be 1 dimensionless group:

$$\Pi_1 = \frac{D}{\rho U^2 d^2} = const., \text{ or } \quad D = (const.)\rho U^2 d^2.$$

Divide both sides of the last equation by $\frac{1}{2}\rho U^2 d^2 \pi/4$ to find:

$$\frac{D}{\frac{1}{2}\rho U^2 d^2 \pi/4} = C_D = \frac{(const.)\rho U^2 d^2}{\frac{1}{2}\rho U^2 d^2 \pi/4} = const.,$$

where the 2, 4, and π have again been absorbed into the undetermined constant.