Stokes Flow

Approximate solutions NS for low Re with exact solutions reduced equations. Similar exact solutions BL equations but counterpoint for very low vs. very high Re.

Stokes flow: $\underline{u} \cdot \nabla\underline{u} \sim 0$ i.e., convective/acceleration negligible

$\nabla \cdot \underline{u} = 0$

$\underline{u}_t = -(1/\rho)\nabla p + \nu\nabla^2\underline{u}$ linear!

More formally using dimensional analysis, which is useful to obtain higher order corrections using matched asymptotic expansions, as follows:

$\underline{u} = U\underline{u}^*$

$p = \frac{\rho\nu U}{l} p^* = \frac{\mu U}{l} p^*$ note pressure scaled by viscous stress $\mu u_y \propto \frac{\mu U}{l}$ vs. ρU^2

$x_i = l x_i^*$

$t = (l^2/\nu)\, t^*$

$U =$ characteristic velocity, e.g., free stream velocity

$l =$ characteristic length, e.g., body dimension

l^2/ν corresponds to time required for viscous diffusion traverse distance l or vice versa $l \propto \sqrt{\nu t}$, as per canonical diffusion layer problems.

Substitution NS

$$\left(\frac{\nu U}{l^2}\right)\underline{u_t^*}+\left(\frac{U^2}{l}\right)\left(\underline{u^*}\cdot\nabla^*\underline{u^*}\right)=-\left(\frac{\nu U}{l^2}\right)\nabla^* p^*+\left(\frac{\nu U}{l^2}\right){\nabla^*}^2\underline{u^*}$$

$$\times\frac{l^2}{\nu U}\qquad \underline{u_t^*}+Re\left(\underline{u^*}\cdot\nabla^*\underline{u^*}\right)=-\nabla^* p^*+{\nabla^*}^2\underline{u^*}\qquad Re\ =\ Ul/\nu$$

$$Re\ \rightarrow\ 0\ drop\ *:\ \underline{u_t}=-\nabla p+\nabla^2\underline{u}$$

Stokes equations asymptotic limit Re → 0 while space coordinates order unity

Stokes equations (dimensional):

$$\nabla\cdot\underline{u}=0\qquad(1)$$

$$\underline{u_t}=-(1/\rho)\nabla p+\nu\nabla^2\underline{u}\qquad(2)$$

$$\nabla\cdot(2)\Rightarrow\nabla^2 p=0\qquad(3)\qquad p\ harmonic$$

$$\nabla\times(2)\Rightarrow\underline{\omega_t}=\nu\nabla^2\underline{\omega}\qquad(4)\qquad\underline{\omega}\ harmonic\ steady\ flow$$

Alternative form (2) without p:

$$\nabla\times\nabla\times(2)\Rightarrow\partial/\partial t\left[\nabla\left(\nabla\cdot\underline{u}\right)-\nabla^2\underline{u}\right]=\nu\nabla^2\left[\nabla\left(\nabla\cdot\underline{u}\right)-\nabla^2\underline{u}\right]$$

$note$:

$$\nabla\times\nabla\times\underline{u}=\nabla(\nabla\cdot u)-\nabla^2\underline{u}$$

$$\nabla\times\nabla p\ =\ 0$$

$$\therefore\ \nabla^2\underline{u_t}=\nu\nabla^4\underline{u}\qquad(5)\ \ p\text{ removed but 4th order}$$

For 2D steady flow, (4) can be expressed in terms of the stream function using $\omega_z=v_x-u_y,\ u=\psi_y,\ v=\psi_x,\ \omega_z=-\nabla^2\psi,$

$$\nabla^4\psi=0\qquad(6)$$

drop $*$ notation $\Rightarrow$

$$\frac{\partial \underline{u}}{\partial t} + Re(\underline{u} \cdot \nabla \underline{u}) = -\nabla p + \nabla^2 \underline{u} \quad (7)$$

$\nabla^2 p$=0

Let

$$\underline{u} = \underline{u}^0 + Re\underline{u}^1 + \cdots \qquad (8)$$

$$p = p^0 + Rep^1$$

By using (8) in (7) and retaining terms of various O(Re), the gde's for $\underline{u}^n$ and p^n can be obtained. For example:

$$\partial \underline{u}^0 / \partial t = -\nabla p^0 + \nabla^2 \underline{u}^0 \qquad O(Re = 0)$$

$$\nabla^2 p^0$$

Note that the equations for the higher-order $\underline{u}^n$ and p^n will include lower-order terms.

Note that Stokes equations are linear which is a great advantage since many methods are available for their solution, e.g.:

- Separation of variables
- Linear superposition of elementary solutions (this approach useful in showing which elements produce $\underline{F}$ and $\underline{M}$)

Types of low Re problems

1. Fully developed laminar duct flows (inertia terms vanish due to geometry assumed)

2. Flows about immersed bodies (microorganisms, etc.)

3. Flows in narrow passages – lubrication theory

4. Flow through porous media $(\nabla^2 \hat{p} = 0)$

Herein only types 2 and 3 flow are considered.

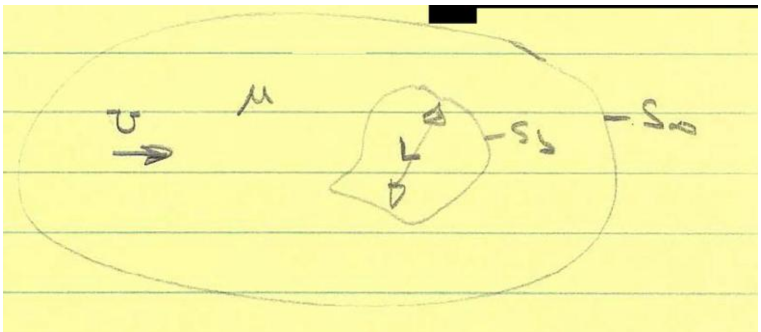
Creeping steady flow about immersed bodies

$$\nabla \cdot \underline{u} = 0$$

$$\nabla p = \mu \nabla^2 \underline{u}$$

$$BC: \underline{u}(S_b) = 0$$

$$\underline{u}(\infty) = U_\infty$$



Stokes Paradox

Assuming inertia/density negligible, force on body only depends on $U, \mu, L.$

2-D body

$F' = F'(U, \mu, L)$ F′ = force per unit length

3-D body

$F' = F'(U, \mu, L)$ F′ = total drag force

Dimensional analysis

2D $\frac{F'}{\mu U} = \text{constant}$

3D $F/(\mu UL) = \text{constant}$

2D unrealistic as does not depend on the size of the body, whereas 3D realistic. ∴ ρ cannot be neglected

2D $F' = F'(U, \mu, L, \rho) \Rightarrow \frac{F'}{\mu U} = F'(Re)$

In the science of fluid flow, **Stokes' paradox** is the phenomenon that there can be no creeping flow of a fluid around a disk in two dimensions; or, equivalently, the fact there is no non-trivial steady-state solution for the Stokes equations around an infinitely long cylinder. This is opposed to the 3-dimensional case, where Stokes' method provides a solution to the problem of flow around a sphere.

Stokes' paradox was resolved by Carl Wilhelm Oseen in 1910, by introducing the Oseen equations which improve upon the Stokes equations – by adding convective acceleration.

Some basic solutions

Since Stokes equations are linear the superposition principle may be applied, and solutions can be obtained by combining basic solutions. Recall that this approach is frequently used in inviscid-flow theory in solving the Laplace equation.

1. Uniform flow

$$\underline{V} = U\hat{\mathrm{e}}_x$$

$$p = \text{constant}$$

Clearly, this velocity & pressure field produces no force or moment. However, as will be shown below, this solution can be combined with others to produce the flow pattern for simple geometries.

2. Doublet

We have discussed the fact that any potential flow is also an exact solution of the full incompressible NS equations since the viscous term is then identically zero. Thus, this is also true for the Stokes equations. However, in this case we also require $\nabla p = 0$, i.e., p = constant. Keep in mind that for a viscous flow the Bernoulli equation is not valid. Recall that $\nabla p = \mu \nabla^2 \underline{u}$, which shows that irrotational solutions Stokes equations require $\nabla p = 0$.

$\nabla^2 \underline{V} = \nabla(\nabla \cdot \underline{V}) - \nabla \times \underline{\omega} = 0$ for incompressible irrotational flow

For irrotational flow,

$\nabla \times \underline{V}=0 \qquad \underline{V} = \nabla\phi = u_x \hat{e}_x + u_r \hat{e}_r$

$\nabla \cdot \underline{V} = 0 \qquad \nabla^2\phi = 0$

Spherical Coordinates (r, θ, ω) $\qquad x = r\cos\theta$

Appendix A $\qquad r = [x^2 + y^2 + z^2]$

$$\phi = A\frac{\cos\theta}{r^2} = Ax/r^3$$

3-D dipole; singular at r = 0; axisymmetric flow.

$$\Rightarrow \quad \left.\begin{aligned} \underline{V} &= A\left(\frac{\hat{e}_x}{r^3} - \frac{3x\hat{e}_r}{r^4}\right) \\ p &= \text{constant} \end{aligned}\right\}$$

Again, no force or moment exerted on fluid.

3. Rotlet

In this case, a solution is sought for which $\underset{\sim}{\omega} \neq 0$ and $p =$ constant. Say,

$$\underline{V} = \underline{r} \times \nabla\phi \qquad (1)$$

$\nabla \cdot (\underline{a} \times \underline{b})$
$= \underline{b} \cdot (\nabla \times \underline{a})$
$- \underline{a} \cdot (\nabla \times \underline{b})$

$$\nabla \cdot \underline{V} = \underbrace{\nabla\phi \cdot (\nabla \times \underline{r})}_{=0} + \underline{r} \cdot (\nabla \times \nabla\phi) = 0$$

=0 by vector identity

$\phi_{,k}$ $\underbrace{\varepsilon_{ijk}}_{alternating\ tensor}$ $\underbrace{x_{k,j}}$ = 0 unless $k = j$, and under this condition $\varepsilon_{ijk} = 0$

$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$

$\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$

$\varepsilon_{ijk} = 0$ otherwise

This shows that the continuity equation is satisfied identically for all forms of ϕ.

Since $\frac{\partial}{\partial t} = 0$ and p = constant, Stokes equation reduces to.

$$\nabla^2 \underline{V} = 0 = \frac{\partial^2 V_i}{\partial x_l \partial x_l}$$

$$= \varepsilon_{ijk} \left[\frac{\partial^2 x_j}{\partial x_l\, \partial x_l} \frac{\partial \phi}{\partial x_k} + x_j \frac{\partial^2 \phi}{\partial x_l\, \partial x_l} \right] = 0$$

$\frac{\partial^2 x_j}{\partial x_l\, \partial x_l} = 0$ since $\frac{\partial x_j}{\partial x_l} = \nabla \cdot \underline{x} = 3$ thus $\frac{\partial^2 \phi}{\partial x_l\, \partial x_l} = 0$

$$\underline{u} = \underline{r} \times \nabla\phi = \varepsilon_{ijk}\, x_j \frac{\partial \phi}{\partial x_k} \qquad c_i = \varepsilon_{ijk}\, a_j b_k \qquad \underline{c} = \underline{a} \times \underline{b}$$

$$\nabla \cdot \underline{u} = \frac{\partial u_i}{\partial x_i} = \varepsilon_{ijk}\left(\underbrace{x_{j,i}\frac{\partial \phi}{\partial x_k}}_{symmetric} + \underbrace{x_j \frac{\partial^2 \phi}{\partial x_i \partial x_k}}_{symmetric}\right)$$ divergence rank r = tensor rank r-1 (here r = 1)

$\varepsilon_{ijk} x_{j,i} = 0$ since $x_{j,i} = 0\ (j \neq i)$ and $\varepsilon_{ijk} = 0\ (j = i)$ for first term

and product antisymmetric & symmetric tensor = 0 both terms

$x_{j,i} = \delta_{ij}$ and $\frac{\partial^2 \phi}{\partial x_i \partial x_k} = \frac{\partial^2 \phi}{\partial x_k \partial x_i}$ are both symmetric

i.e. $\nabla \cdot u = 0$ Alternative derivation $\nabla \cdot u = \nabla\phi \cdot (\nabla \times \underline{r}) + \underline{r} \cdot (\nabla \times \nabla\phi) = 0$

$\nabla p = \nabla^2 \underline{u}$, but p = constant $\Rightarrow \nabla^2 \underline{u} = 0$

$$\nabla^2 u_i = \frac{\partial^2 u_i}{\partial x_l \partial x_l} = \varepsilon_{ijk}\left[\underbrace{\frac{\partial^2 x_j}{\partial x_l \partial x_l}\frac{\partial \phi}{\partial x_k}}_{(1)=0} + \underbrace{x_j \frac{\partial}{\partial x_k}\frac{\partial^2 \phi}{\partial x_l \partial x_l}}_{(2)=0}\right] = \nabla^2 \underline{u} = 0$$

$$\frac{\partial}{\partial x_l}\left(\frac{\partial x_j}{\partial x_l}\right) = \nabla \cdot (\nabla \cdot \underline{x}) = \nabla \cdot (3) = 0$$

Therefore $\underline{u} = \underline{r} \times \nabla\phi$ satisfies the continuity and Stokes equations

for p = constant *assuming* $\nabla^2 \phi = 0$, $\qquad (\underline{a} \cdot \nabla)\underline{x} = \underline{a}$

therefore need axisymmetric solutions $\nabla^2 \phi = 0$ $\qquad \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$

Alternatively $\qquad \nabla \times (\nabla \times \underline{u}) = \nabla(\nabla \cdot \underline{u}) - \nabla^2 \underline{u} \qquad \nabla^2 \underline{u} = -\nabla \times (\nabla \times \underline{u})$

$$\nabla \times \underline{u} = \nabla \times (\underline{r} \times \nabla\phi) = \underline{r}\overbrace{(\nabla \cdot \nabla\phi)}^{\nabla^2\phi = 0} + \overbrace{(\nabla\phi \cdot \nabla)\underline{r}}^{\nabla\phi} - \nabla\phi\left(\underbrace{\nabla \cdot \underline{r}}_{3}\right) - (\underline{r} \cdot \nabla)\nabla\phi$$

$$\left((\underline{r} \cdot \nabla)\nabla\phi = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\nabla\phi = \nabla\psi\right) \quad \psi = r \cdot \nabla\phi - \phi$$

$$\nabla \times (\nabla f) = 0$$

The following shows the steps to show that $(\underline{r} \cdot \nabla)\nabla\phi = \nabla\psi$ writing each term by components.

Need to prove that:

$$(r \cdot \nabla)\nabla\phi = \nabla\psi$$

Write $(r \cdot \nabla)\nabla\phi$ by components:

$$(r\cdot\nabla)\nabla\phi = \left(r_1\frac{\partial}{\partial r_1} + r_2\frac{\partial}{\partial r_2} + r_3\frac{\partial}{\partial r_3}\right)\begin{pmatrix}\frac{\partial\phi}{\partial r_1}\\ \frac{\partial\phi}{\partial r_2}\\ \frac{\partial\phi}{\partial r_3}\end{pmatrix} = \begin{pmatrix} r_1\frac{\partial^2\phi}{\partial r_1^2} + r_2\frac{\partial^2\phi}{\partial r_1\partial r_2} + r_3\frac{\partial^2\phi}{\partial r_1\partial r_3}\\ r_1\frac{\partial^2\phi}{\partial r_2\partial r_1} + r_2\frac{\partial^2\phi}{\partial r_2^2} + r_3\frac{\partial^2\phi}{\partial r_2\partial r_3}\\ r_1\frac{\partial^2\phi}{\partial r_3\partial r_1} + r_2\frac{\partial^2\phi}{\partial r_3\partial r_2} + r_3\frac{\partial^2\phi}{\partial r_3^2}\end{pmatrix}$$

Take the elements in the first row:

$$r_1\frac{\partial^2\phi}{\partial r_1^2} = \frac{\partial}{\partial r_1}\left(r_1\frac{\partial\phi}{\partial r_1}\right) - \frac{\partial r_1}{\partial r_1}\frac{\partial\phi}{\partial r_1} = \frac{\partial}{\partial r_1}\left(r_1\frac{\partial\phi}{\partial r_1}\right) - \frac{\partial\phi}{\partial r_1}$$

$$r_2\frac{\partial^2\phi}{\partial r_1\partial r_2} = \frac{\partial}{\partial r_1}\left(r_2\frac{\partial\phi}{\partial r_2}\right) - \frac{\partial r_2}{\partial r_1}\frac{\partial\phi}{\partial r_2} = \frac{\partial}{\partial r_1}\left(r_2\frac{\partial\phi}{\partial r_2}\right)$$

$$r_3\frac{\partial^2\phi}{\partial r_1\partial r_3} = \frac{\partial}{\partial r_1}\left(r_3\frac{\partial\phi}{\partial r_3}\right) - \frac{\partial r_3}{\partial r_1}\frac{\partial\phi}{\partial r_3} = \frac{\partial}{\partial r_1}\left(r_3\frac{\partial\phi}{\partial r_3}\right)$$

Where the following rule has been applied:

$$\frac{\partial r_i}{\partial r_j} = \delta_{ij}$$

Doing this process for all terms gives:

$$(r \cdot \nabla)\nabla\phi = \begin{pmatrix} \frac{\partial}{\partial r_1}\left(r_1\frac{\partial\phi}{\partial r_1}\right) - \frac{\partial\phi}{\partial r_1} + \frac{\partial}{\partial r_1}\left(r_2\frac{\partial\phi}{\partial r_2}\right) + \frac{\partial}{\partial r_1}\left(r_3\frac{\partial\phi}{\partial r_3}\right)\\ \frac{\partial}{\partial r_2}\left(r_1\frac{\partial\phi}{\partial r_1}\right) + \frac{\partial}{\partial r_2}\left(r_2\frac{\partial\phi}{\partial r_2}\right) - \frac{\partial\phi}{\partial r_2} + \frac{\partial}{\partial r_2}\left(r_3\frac{\partial\phi}{\partial r_3}\right)\\ \frac{\partial}{\partial r_3}\left(r_1\frac{\partial\phi}{\partial r_1}\right) + \frac{\partial}{\partial r_3}\left(r_2\frac{\partial\phi}{\partial r_2}\right) + \frac{\partial}{\partial r_3}\left(r_3\frac{\partial\phi}{\partial r_3}\right) - \frac{\partial\phi}{\partial r_3}\end{pmatrix}$$

Group the $\frac{\partial}{\partial r_i}$ terms:

$$(r \cdot \nabla)\nabla\phi = \begin{pmatrix} \frac{\partial}{\partial r_1}\left(r_1\frac{\partial\phi}{\partial r_1} + r_2\frac{\partial\phi}{\partial r_2} + r_3\frac{\partial\phi}{\partial r_3}\right) - \frac{\partial\phi}{\partial r_1}\\ \frac{\partial}{\partial r_2}\left(r_1\frac{\partial\phi}{\partial r_1} + r_2\frac{\partial\phi}{\partial r_2} + r_3\frac{\partial\phi}{\partial r_3}\right) - \frac{\partial\phi}{\partial r_2}\\ \frac{\partial}{\partial r_3}\left(r_1\frac{\partial\phi}{\partial r_1} + r_2\frac{\partial\phi}{\partial r_2} + r_3\frac{\partial\phi}{\partial r_3}\right) - \frac{\partial\phi}{\partial r_3}\end{pmatrix}$$

Rewrite as dot product of two vectors:

$$(r \cdot \nabla)\nabla\phi = \begin{pmatrix}\frac{\partial}{\partial r_1}(r\cdot\nabla\phi) - \frac{\partial\phi}{\partial r_1}\\ \frac{\partial}{\partial r_2}(r\cdot\nabla\phi) - \frac{\partial\phi}{\partial r_2}\\ \frac{\partial}{\partial r_3}(r\cdot\nabla\phi) - \frac{\partial\phi}{\partial r_3}\end{pmatrix} = \begin{pmatrix}\frac{\partial}{\partial r_1}(r\cdot\nabla\phi - \phi)\\ \frac{\partial}{\partial r_2}(r\cdot\nabla\phi - \phi)\\ \frac{\partial}{\partial r_3}(r\cdot\nabla\phi - \phi)\end{pmatrix} = \nabla(r\cdot\nabla\phi - \phi) = \nabla\psi$$

$\therefore$ if $\nabla^2\phi = 0$ Stokes equation is satisfied. The problem again reduces to that of obtaining solutions to Laplace equation. Here we are interested in solutions to $\nabla^2\phi$ for axisymmetric flow. Recall from inviscid-flow theory that the solutions are in terms of spherical harmonics. The first harmonic (source) is

$$\phi \sim \frac{1}{r} \qquad \underline{V} = \underline{r} \times \nabla\phi$$

$$\Rightarrow \nabla\phi \propto \widehat{e_r} \text{ so that } \underline{V} \propto \underbrace{r\widehat{e_r}}_{\underline{r}} \times \widehat{e_r} = 0$$

The next solution (doublet) is

$$\widehat{e_r} = \cos\theta\ \widehat{e_x} + \sin\theta\cos\omega\ \widehat{e_y} + \sin\theta\sin\omega\ \widehat{e_z}$$

$$\phi = B\frac{\cos\theta}{r^2} = \frac{Bx}{r^3}$$

$$\underline{V} = \underline{r} \times \nabla\left(\frac{Bx}{r^3}\right)$$

$$= B\,\underline{r} \times \left(\frac{\widehat{e_x}}{r^3} - \frac{3x\,\widehat{e_r}}{r^4}\right)$$

$$= \frac{B}{r^2}\ \widehat{e_r} \times \widehat{e_x}$$

Streamlines perpendicular $\widehat{e_r}$ *and* $\widehat{e_x}$, i.e., circles whose centers lie on x-axis.

Where in this case $\nabla = \frac{\partial}{\partial x}\widehat{e_x} + \frac{\partial}{\partial r}\widehat{e_r} \quad \underline{V} = f(x, r)$

Also $\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \quad \underline{r} = r\widehat{e_r} \qquad \widehat{e_r} \times \widehat{e_r} = 0$

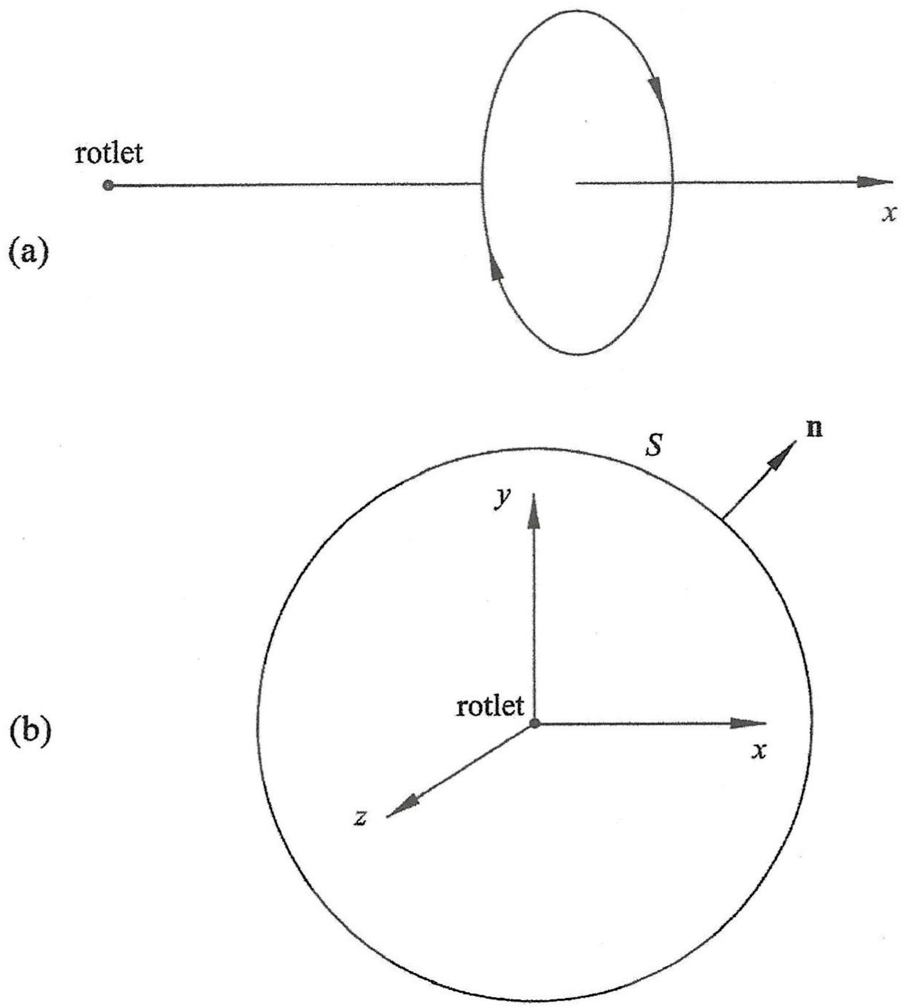


FIGURE 8.1 (a) Typical streamline due to a rotlet, and (b) spherical control surface surrounding a rotlet.

A rotlet exerts a moment but not a force on the fluid

$$F_i = -\int_S \sigma_{ij} n_j \, dS = -\int_S \left[-p\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right)\right] n_j \, dS$$

(S = sphere about rotlet) since p=constant, $n_j = \frac{x_j}{r}$

$$u_{i,j} \sim r^{-3}, \quad dS \sim r^2 \;\Rightarrow\; F_i \sim \frac{1}{r^3} r^2 = \frac{1}{r}\Big|_{r\to\infty} = 0$$

$$\Rightarrow F_i = 0$$

$$\underline{M} = \int_S \underline{r} \times \underline{P}\, dS, \quad \underline{P} = \text{surface force vector}$$

$$P_k = \sigma_{kl} n_l$$

$$M_i = \int_S \varepsilon_{ijk} x_j \sigma_{kl} n_l \; dS \qquad n_l = \frac{x_l}{r}$$

which can be evaluated to yield $\underline{M} = 8\pi\mu B\, \widehat{e_x}$, as shown next.

$$M_i = \frac{\mu}{r} \int_S \varepsilon_{ijk} x_j x_l \left(u_{k,l} + u_{l,k}\right) dS \qquad \varepsilon_{ijk}\, p\delta_{ij} = 0$$

$$\widehat{e_r} = \cos\theta\ \hat{\imath} + \sin\theta \cos\omega\ \hat{\jmath} + \sin\theta \sin\omega\ \hat{k}$$

$$\widehat{e_r} \times \widehat{e_x} = \sin\theta \cos\omega\ \hat{k} - \sin\theta \sin\omega\ \hat{\jmath} = \widehat{e_u}$$

$$\underline{u} = \frac{B}{r^2}\widehat{e_u} \quad \frac{\partial u_i}{\partial r} = -2Br^{-3} = -\frac{2B}{r^3} \qquad \frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

$$u_{1,1} = -\frac{2B}{r^3}\frac{x}{r} \qquad r = (x^2 + y^2 + z^2)^{1/2}$$

$$r_x = \frac{1}{2}[\quad]^{-1/2} 2x = x/[\quad]^{1/2} = x/r$$

$$-\frac{2B}{r^4}\Bigg(\underbrace{x^2 + y^2 + z^2}_{r^2}\Bigg) = x_l u_{1,l} = -\frac{2B}{r^2} = -2u_1$$

$\therefore x_l u_{k,l} = -2u_k$

Shows $\underline{u} = \frac{B}{r^2}\widehat{e_u}$ homogeneous function degree 2 https://en.wikipedia.org/wiki/Homogeneous_polynomial, which satisfies $f\left(\frac{x}{\lambda}, \frac{y}{\lambda}, \frac{z}{\lambda}\right) = \lambda^m f(x, y, z)$ and for such functions

Euler theorem states that $x f_x + y f_y + z f_z = -\text{mf}$

$$x_l u_{l,k} = \underbrace{\frac{\partial}{\partial x_k}(x_l u_l)} - u_l \overbrace{\frac{\partial x_l}{\partial x_k}}^{x_{l,k}=\delta_{lk}} = -u_l \delta_{lk} = -u_k$$

$$u_l x_{l,k} + x_l u_{l,k} = \nabla(\underline{r} \cdot \underline{u}) = 0 \qquad \underline{r} \perp \underline{u}$$

Shows $x_l u_{l,k} = -u_k$ since $\underline{u} = B \frac{\widehat{e_r} \times \widehat{e_x}}{r^2}$ and $\underline{u} \perp \widehat{e_r}$ and $\widehat{e_x}$

$$M_i = \frac{\mu}{r} \int_S \varepsilon_{ijk} x_j (-2u_k - u_k)\ dS$$

$$\underline{M} = -\frac{3\mu}{r} \int_S \underline{r} \times \underline{u}\, dS$$

$$\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\widehat{e_r} = a\hat{\imath} + b\hat{\jmath} + c\hat{k} = \cos\theta\,\hat{\imath} + \sin\theta\cos\omega\,\hat{\jmath} + \sin\theta\sin\omega\,\hat{k}$$

$$\widehat{e_r} \times \widehat{e_x} = +c\hat{\jmath} - b\hat{k} \qquad (\mathrm{a}\,\hat{\imath} + \mathrm{b}\,\hat{\jmath} + \mathrm{c}\,\hat{k}) \times \hat{\imath} = -b\hat{k} + c\hat{\jmath}$$

$$\underline{r} \times (\widehat{e_r} \times \widehat{e_x}) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}) \times (+c\hat{\jmath} - b\hat{k})$$

$$= +xc\hat{k} + xb\hat{\jmath} - yb\hat{\imath} - zc\hat{\imath}$$

$$= -(yb + zc)\hat{\imath} + xb\hat{\jmath} + xc\hat{k}$$

$$\underline{r} \times (\widehat{e_r} \times \widehat{e_x}) = (\underline{r} \cdot \widehat{e_x})\widehat{e_r} - (\underline{r} \cdot \widehat{e_r})\widehat{e_x}$$

$$\mathrm{x}(a\hat{\imath} + b\hat{\jmath} + c\hat{k}) - (xa + yb + zc)\hat{\imath}$$

$$= (xa - xa - yb - zc)\hat{\imath} + xb\hat{\jmath} + xc\hat{k}$$

$$= -(yb + zc)\hat{\imath} + xb\hat{\jmath} + xc\hat{k} - (yb + zc)\hat{\imath} + \underbrace{xa\hat{\imath} + xb\hat{\jmath} + xc\hat{k}}_{x\widehat{e_r}} - xa\hat{\imath}$$

$$= x\widehat{e_r} + (-xa - yb - zc)\hat{\imath} = x\widehat{e_r} - (xa + yb + zc)\hat{\imath} = x\widehat{e_r} - r\hat{k}$$

$$-(yb+cz)\hat{\imath} + \underbrace{xb\hat{\jmath} + xc\hat{k} + xa\hat{\imath}}_{+x\widehat{e_r}} - xa\hat{\imath}$$

$$+x\widehat{e_r} - (xa+yb+zc)\hat{\imath} \qquad \underline{r}\cdot\widehat{e_r} = \frac{\underline{r}\cdot\underline{r}}{|r|} = r$$

$$+x\widehat{e_r} - r\hat{\imath}$$

$$\underline{r}\times\underline{u} = \frac{B}{r^2}\underbrace{\underline{r}\times(\widehat{e_r}\times\widehat{e_x})}_{(\underline{r}\cdot\widehat{e_x})\widehat{e_r}-(\underline{r}\cdot\widehat{e_r})\widehat{e_x}} = \frac{B}{r^2}[x\widehat{e_r} - r\widehat{e_x}]$$

$$-\frac{3\mu}{r}\left(\frac{Bx}{r^2}\widehat{e_r} - \frac{B}{r}\widehat{e_x}\right) = \frac{3B\mu}{r^2}\left(\frac{x}{r}\widehat{e_r} - \widehat{e_x}\right) \qquad \underline{M} = -3B\mu\int_S\left(\frac{x}{r}\widehat{e_r} - \widehat{e_x}\right)\frac{dS}{r^2}$$

Note: $\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$

$$x = r\cos\theta, \qquad dS = r^2\sin\theta\ d\theta d\omega$$

$$\underline{M} = -3B\mu\int_0^{2\pi}d\omega\int_0^{\pi}\left[\cos\theta\left(a\hat{\imath} + b\hat{\jmath} + c\hat{k}\right) - \hat{\imath}\right]\sin\theta\ d\theta$$

$$\cos\theta\left(\cos\theta\hat{\imath} + \sin\theta\cos\omega\hat{\jmath} + \sin\theta\sin\omega\hat{k}\right) - \hat{\imath}$$

$$\underline{M} = -3B\mu\int_0^{2\pi}d\omega\int_0^{\pi}\left[(\cos^2\theta - 1)\hat{\imath} + \cos\theta\sin\theta\cos\omega\hat{\jmath} + \cos\theta\sin\theta\sin\omega\hat{k}\right]\sin\theta\ d\theta$$

$M_i = 8\pi B\mu\,\widehat{e_x}$ rotlet $F_i = 0 \quad M_i \neq 0$

$\propto B = velocity\ magnitude$ $\underline{u} = B\frac{\widehat{e_r}\times\widehat{e_x}}{r^2}$

right hand rule in x direction $p = \text{constant}$

Euler's Homogeneous Function Theorem

Let $f(x, y)$ be a homogeneous function of order n so that

$$f(t x, t y) = t^n f(x, y). \tag{1}$$

Then define $x' \equiv x t$ and $y' \equiv y t$. Then

$$n t^{n-1} f(x, y) = \frac{\partial f}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial t} \tag{2}$$

$$= x \frac{\partial f}{\partial x'} + y \frac{\partial f}{\partial y'} \tag{3}$$

$$= x \frac{\partial f}{\partial (x t)} + y \frac{\partial f}{\partial (y t)}. \tag{4}$$

Let $t = 1$, then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x, y). \tag{5}$$

This can be generalized to an arbitrary number of variables

$$x_i \frac{\partial f}{\partial x_i} = n f(\mathbf{x}), \tag{6}$$

where Einstein summation has been used.

CITE THIS AS:

Weisstein, Eric W. "Euler's Homogeneous Function Theorem." From *MathWorld*--A Wolfram Web Resource. https://mathworld.wolfram.com/EulersHomogeneousFunctionTheorem.html

SUBJECT CLASSIFICATIONS

Calculus and Analysis › Functions ›

4. Stokeslet

Now we seek a solution for which p ≠ constant. Since the gde for p is the Laplace equation we can again immediately write down solutions

$p \propto 1/r$ (source)

$p \propto cos\theta/r^2$ (dipole)

etc.

The source solution turns out not to be of interest, but the dipole one leads to the Stokeslet.

$p = 2c\mu x/r^3$ $x = rcos\theta$ and constant = $2c\mu$

Then, from Stokes equation

$$\nabla^2\underline{V} = \frac{1}{\mu}\nabla p$$

or

$$\nabla^2 u = c\left(\frac{2}{r^3} - 6\frac{x^2}{r^5}\right)$$

$$\nabla^2 v = -6c\frac{xy}{r^5}$$

$$\nabla^2 w = -6c\frac{xz}{r^5}$$

$$u_{xx} + u_{yy} + u_{zz} = (1/\mu)p_x = (1/\mu)\left[2c\mu\frac{\partial}{\partial x}(x/r^3)\right] \qquad \partial r/\partial x_i = x_i / r$$

$$= 2c[r^{-3} + x(-3r^{-4} \cdot x/r)]$$

$$= 2c[r^{-3} - 3x^2/r^5]$$

$$\therefore \nabla^2 u = c(2/r^3 - 6x^2/r^5)$$

$$\nabla^2 v = \frac{1}{\mu}p_y = \frac{1}{\mu}[2c\mu x(-3r^{-4}y/r)]$$

$$= 2cx(-3y/r^5)$$

$$\nabla^2 v = -6cxy/r^5$$

$$\nabla^2 \omega = \frac{1}{\mu}p_z = -6cxz/r^5$$

The particular integrals to these nonhomogeneous pde's can be obtained using the properties of harmonic functions and the following identities:

$$\nabla^2(\tfrac{1}{r^n}) = \frac{n(n-1)}{r^{n+2}} \qquad r^2 = x^2 + y^2 + z^2$$

$$\nabla^2(\varphi\psi) = \psi\nabla^2\phi + \phi\nabla^2\psi + 2\nabla\phi \cdot \nabla\psi$$

$$\Rightarrow u = c\frac{x^2}{r^3} \qquad \phi = r^{-3} \qquad \psi = x^2$$

$$v = c\frac{xy}{r^3} \qquad \phi = r^{-3} \qquad \psi = xy$$

$$w = c\frac{xz}{r^3} \qquad \phi = r^{-3} \qquad \psi = xz$$

For example:

$\nabla^2 v = -6cxy/r^5$

$\phi = r^{-3}, \qquad \psi = xy$

$\nabla^2\phi = 6/r^5$

$\nabla\phi = -3/r^4\,\hat{e}_r$

$\nabla^2\psi = 0$

$\nabla\psi = y\hat{e}_x + x\hat{e}_y$

$\nabla\phi\cdot\nabla\psi = -6xy/r^5$

$\nabla^2(\phi\psi) = \nabla^2(xy/r^3) = -6xy/r^5$

$\Rightarrow v = cxy/r^3$

x = r cosθ

y = r sinθ cosω

z = r sinθ sinω

$\underline{V} = (u, v, w) + \underline{V}'$ where V′ = solution to the homogeneous equation $\nabla^2\underline{V} = 0$

$= c\frac{x}{r^2}\hat{e}_r + \underline{V}'$

$\hat{e}_r = cos\theta\hat{e}_x + sin\theta cos\omega\hat{e}_y + sin\theta sin\omega\hat{e}_z$

Next, $\underline{V}'$ is determined from the continuity equation

$$\nabla \cdot \underline{V} = 0$$

$$= c\nabla \cdot \left(\frac{x}{r^2}\hat{e}_r\right) + \nabla \cdot \underline{V}'$$

$$\nabla \cdot \left(\frac{x}{r^2}\right)\hat{e}_r = \nabla \cdot \left(\frac{x}{r^3}\underline{r}\right)$$

$$= \underline{r} \cdot \nabla\left(\frac{x}{r^3}\right) + \frac{x}{r^3}\nabla \cdot \underline{r} \qquad \nabla \cdot \underline{r} = 3$$

$$= \left[x\left(\frac{1}{r^3} - 3\frac{x^2}{r^5}\right) - y\frac{3xy}{r^5} - z\frac{3xz}{r^5}\right] + 3\frac{x}{r^3}$$

$$= \frac{x}{r^3}$$

$$\nabla \cdot \underline{V} = cx/r^3 + \nabla \cdot \underline{V}' = 0$$

$$\Rightarrow \underline{V}' = c\frac{\hat{e}_x}{r} \qquad \text{note that } \nabla^2 V' = 0$$

So that the final solution is

$$\left.\begin{aligned} p &= 2c\mu x/r^3 \\ \underline{V} &= c\left(\frac{x}{r^2}\hat{e}_r + \frac{1}{r}\hat{e}_x\right)\end{aligned}\right\}\text{Stokeslet again, singular at origin}$$

In this case, the force is nonzero but there is no moment

$$\underline{F} = 8\pi c\mu\hat{e}_x$$

$$\underline{M} = 0$$

$F_i = -\int_S \sigma_{ij} n_j \; dS$ force on the fluid

$\sigma_{ij} = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i})$

$F_i = -\int_S \left[-\frac{2c\mu x_k}{r^3}\delta_{ij} + \mu(u_{i,j} + u_{j,i})\right] n_j \; dS \qquad n_j = \frac{x_j}{r}$

$$= -\int_S \left[-\frac{2c\mu x}{r^3}\frac{x_i}{r} + \frac{\mu}{r} x_j (u_{i,j} + u_{j,i})\right] dS$$

$= 6c\mu \int_S \frac{x x_i}{r^4} \; dS \quad$ x$(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$

x$(r\cos\theta\hat{\imath} + r\sin\theta\cos\phi\hat{\jmath} + r\sin\theta\sin\phi\hat{k})$

Substituting

x $r\widehat{e_r}$

$x = r\cos\theta$

$\widehat{e_r} = cos\,\omega\hat{\imath} + sin\,\theta\,cos\,\omega\hat{\jmath} + sin\,\theta\,sin\,\omega\hat{k}$

$dS = r^2\sin\theta \; d\theta\, d\omega$

$$\underline{F} = 6c\mu \int_S \frac{x}{r^3}\widehat{e_r}\, dS$$

$$\underline{F} = 6c\mu \int_0^{2\pi} d\omega \int_0^{\pi} \cos\theta \left(\cos\theta\widehat{e_x} + \sin\theta\cos\omega\widehat{e_y} + \sin\theta\sin\omega\widehat{e_z}\right)\sin\theta \; d\theta$$

$\underline{F} = 8\pi\mu\,\widehat{e_x} \;\propto c$

direction $+x$ if $c > 0$

$$\underline{u} = c\left[\frac{x}{r^2}\left(cos\theta\hat{e}_x + sin\theta cos\omega\hat{e}_y + sin\theta sin\omega\hat{e}_z\right) + \frac{\hat{e}_x}{r}\right]$$

$$= c\left[(xcos\theta/r^2 + 1/r)\hat{e}_x + \frac{x}{r^2}\left(sin\theta cos\omega\hat{e}_y + sin\theta sin\omega\hat{e}_z\right)\right]$$

$x = r\cos\theta$

$\cos\theta = \frac{x}{r}$

$y = r\sin\theta\cos\omega$

$\frac{y}{r} = \sin\theta\cos\omega$

$z = r\sin\theta\sin\omega$

$\frac{z}{r} = \sin\theta\sin\omega$

$\underline{u} = c\left(\frac{x}{r^2}\widehat{e_r} + \frac{\widehat{e_x}}{r}\right)$

$\widehat{e_r} = \cos\theta\ \widehat{e_x} + \sin\theta\cos\omega\ \widehat{e_y} + \sin\theta\sin\omega\ \widehat{e_z}$

$$-\underline{u} = -c\left[\left(\frac{x^2}{r^3} + \frac{1}{r}\right)\widehat{e_x} + \frac{xy}{r^3}\widehat{e_y} + \frac{xz}{r^3}\widehat{e_z}\right]$$

$$-u_i = -c\left[\frac{x}{r^3}x_i + \frac{\delta_{i1}}{r}\right] = x_j\frac{\partial u_i}{\partial x_j}$$

Shows $\underline{u} = c\left(\frac{x}{r^2}\hat{e}_r + \frac{1}{r}\hat{e}_x\right)$ homogeneous degree 1 and follows Euler theorem. See Appendix B

$$x_j u_{j,i} = \underbrace{\frac{\partial}{\partial x_i}\left(x_j u_j\right)}_{\frac{\partial x_j}{\partial x_i}u_j + x_j\frac{\partial u_j}{\partial x_i}} - u_j\frac{\partial x_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\underline{r}\cdot\underline{u}\right) - u_i$$

$$u_j\frac{\partial x_j}{\partial x_i} = u_j\delta_{ij} = u_i$$

$$\underline{r}\cdot\underline{u} = r\widehat{e_r}\cdot\underline{u} = \frac{cx}{r} + c\cos\theta = \frac{2cx}{r}$$

$$\widehat{e_r} = \frac{\underline{r}}{|\underline{r}|} = \frac{\underline{r}}{r} \qquad r\widehat{e_r} = \underline{r}$$

$$\frac{\partial}{\partial x_i}\left(\frac{2cx}{r}\right) - c\left[\frac{x}{r^3}x_i + \frac{\delta_{i1}}{r}\right] = \frac{\partial}{\partial x_i}\left(\underline{r}\cdot\underline{u}\right) - u_i \qquad \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = -\frac{x_i}{r^3}$$

$$= c\left(\frac{2}{r}\delta_{i1} - \frac{2xx_i}{r^3}\right) - c\left[\frac{x}{r^3}x_i + \frac{\delta_{i1}}{r}\right]$$

$$= c\left(\frac{\delta_{i1}}{r} - \frac{3xx_i}{r^3}\right)$$

$$F_i = -\int_S\left[-\frac{2c\mu x}{r^3}\frac{x_i}{r} - \frac{\mu}{r}c\left(\frac{x}{r^3}x_i + \frac{\delta_{i1}}{r}\right) + \frac{\mu}{r}c\left(-\frac{3xx_i}{r^3} + \frac{\delta_{i1}}{r}\right)\right]dS$$

$$= 6c\mu\int_S\frac{xx_i}{r^4}\,dS$$

Rotating Sphere

One might anticipate that the solution for a sphere rotating with constant angular velocity Ω about the x-axis can be obtained from a rotlet with its strength adjusted to satisfy the no-slip condition.

$$\underline{V} = B\,\frac{(\hat{e}_r \times \hat{e}_x)}{r^2}$$

$$BC\colon \underline{V}(\infty) = 0, \underline{V}(r = a) = \Omega a \hat{e}_r \times \hat{e}_x$$

$$\Rightarrow B = \Omega a^3$$

$$\Rightarrow \underline{V} = \frac{\Omega a^3}{r^2}\hat{e}_r \times \hat{e}_x$$

$$\underline{M} = -8\pi\mu\Omega a^3 \hat{e}_x$$

moment on sphere (acts in a direction which opposes the motion of the sphere)

Uniform flow past an immersed sphere (Appendix C Stokes stream function and Appendix D potential flow solution).

The solution corresponding to uniform flow past a sphere can be obtained by superimposing the three solutions for uniform flow, a doublet, and a Stokeslet. Alternatively, the solution can be obtained using the stream function in spherical coordinates and the technique of separation of variables, as per Chapters 3 (9.2) and 3 (9.3).

$$\underline{V} = U\widehat{e_x} + A\left(\frac{\widehat{e_x}}{r^3} - \frac{3x\widehat{e_r}}{r^4}\right) + c\left(\frac{x}{r^2}\widehat{e_r} + \frac{\widehat{e_x}}{r}\right)$$

$$p = 2c\mu x/r^3$$

$$BC: \qquad \underline{V}(\infty) = U\hat{e}_x, \qquad \underline{V}(a) = 0$$

The condition at ∞ is automatically satisfied. The two constants A and c are now determined to impose the no-slip condition.

$$\underline{V}(x = r = a) = 0$$

$$= U\widehat{e_x} + A\left(\frac{\widehat{e_x}}{a^3} - \frac{3\widehat{e_r}}{a^3}\right) + c\left(\frac{x}{a}\widehat{e_r} + \frac{\widehat{e_x}}{a}\right)$$

$$\Rightarrow 0 = U + A/a^3 + c/a$$

$$0 = -3A/a^3 + c/a$$

$$A = -Ua^3/4 \qquad c = -(3/4)Ua$$

$$\underline{V} = U\left[\widehat{e_x} - \frac{1}{4}\frac{a}{r}\left(\frac{a^2}{r^2} + 3\right)\widehat{e_x} + \frac{3}{4}\frac{ax}{r^2}\left(\frac{a^2}{r^2} - 1\right)\widehat{e_r}\right]$$

$$p = p_\infty - \frac{3}{2}\mu U\frac{ax}{r^3} = p_\infty - \frac{3\mu a U}{2r^2}\cos\theta \qquad \propto \mu\text{, antisymmetric + front – rear}$$

Also, see Chapters 3 (9.2) and 3 (9.3) for ψ and $\underline{V} = (u_r, u_\theta)$.

- velocity field independent of μ
- velocity field symmetric (since convection neglected)
- near sphere $|V| < U$ (i.e., no high velocity region near shoulder as indicated by inviscid solution)
- effect of sphere extends to large distance: at $r = 10a$ the velocities are still 10% below their free-stream values

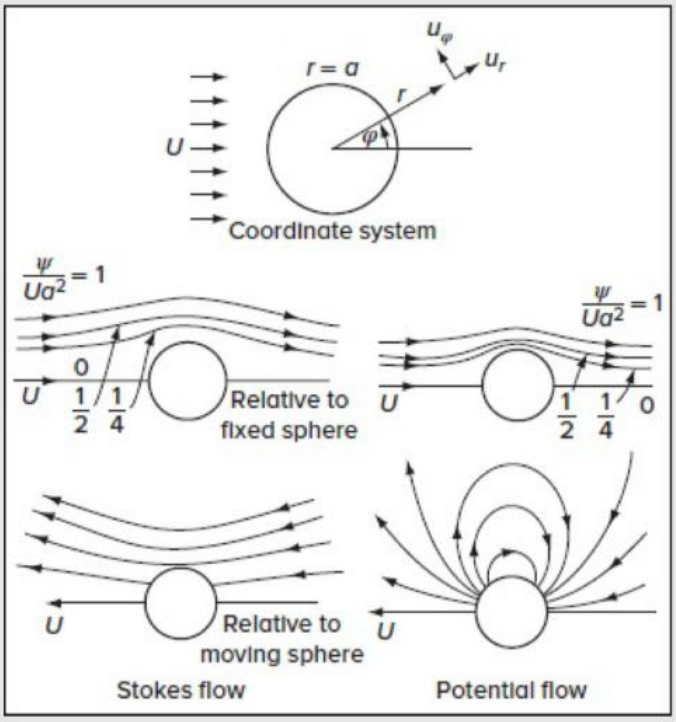


FIGURE 3-42

Comparison of creeping flow (*left*) and potential flow (*right*) past a sphere.

The force on the sphere is

$F = 6\pi\mu U a \hat{e}_x$ Stokes drag law

$F \propto \mu U L$ as anticipated from dimensional analysis

$F = \frac{2}{3}$ friction $+ \frac{1}{3}$ pressure Chapters 3 (9.2) and 3 (9.3)

$\Rightarrow C_D = \frac{2F}{\rho U^2 \pi a^2} = \frac{24}{Re}$ $\quad Re = \frac{2aU}{\nu}$ $\quad Re \ll 1$

This formula is one of the few analytic drag formulas available and the only one for a sphere out of the entire Re range!

Also, valid for Re $\lesssim$ 1.

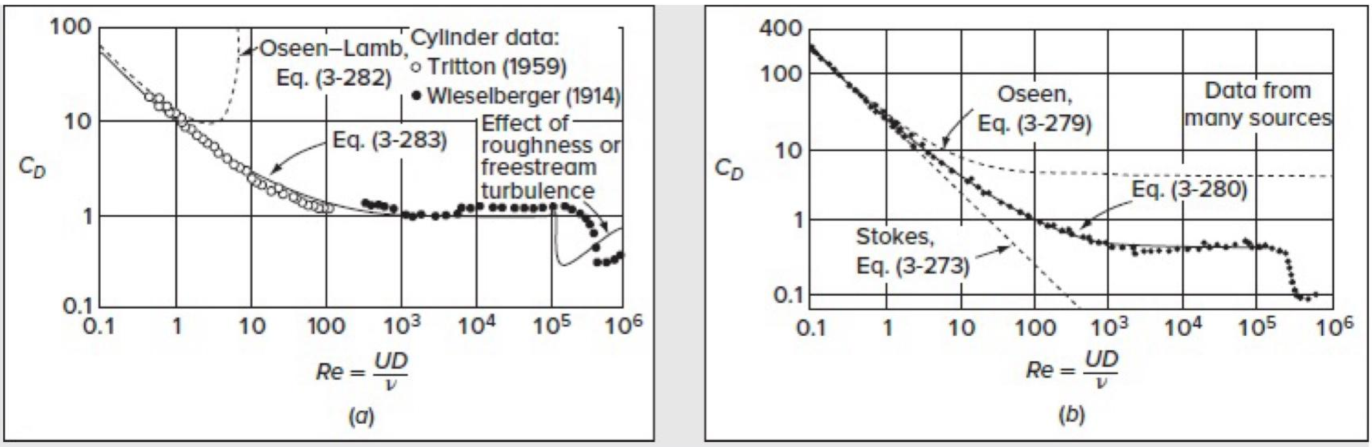


FIGURE 3-45

Comparison of experiment, theory, and empirical formulas for drag coefficients of a cylinder and a sphere (with smooth walls): (*a*) cylinder; (*b*) sphere.

Two other non-spherical solutions of interest are for a circular disk:

normal to freestream: $F = 16\,\mu U a$ (~15% sphere)

parallel to freestream: $F = \frac{32}{3}\mu U a$ (~43% sphere)

⇒ Stokes sphere law is also approximately valid for non-spherical bodies and is often used to estimate drag for roughly spherical bodies (sand grains, dust, etc.).

Two-dimensional creeping flow & Oseen's improvement

Previously Stokes paradox was mentioned. We now show this more formally and outline the procedure for its remedy.

We now attempt to solve the problem of 2-D flow around a circular cylinder. In this case, the technique of separation of variables is used since we have not established any basic solutions to Stokes equations for 2-D flow. Beginning with the vorticity-transport equation in terms of ψ:

$$\nabla^4\psi = 0$$

where for polar coordinates (r, θ)

$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)$ and $v_r = \frac{1}{r}\psi_\theta$ $v_\theta = -\psi_r$

Try $\psi(r,\theta) = f(r)\sin\theta$

(since $\psi(\infty)$ = Ur $\sin\theta$ is uniform flow)

$$\Rightarrow f(r) = Ar^3 + Brlogr + cr + D/r$$

from $\psi(\infty,\theta) = Ursin\theta \Rightarrow A = B = 0, c = U$

$$\psi(r,\theta) = (Ur + D/r)sin\theta$$

on $r = a$: $\psi_\theta = \psi_r = 0$ (no-slip condition)

$v_\theta = -\psi_r = -\left(U - {}^{D}/_{r^2}\right)sin\theta$ = 0 for r=a $\Rightarrow D = Ua^2$

$v_r = \frac{1}{r}\psi_\theta = \left(U + {}^{D}/_{r^2}\right)cos\theta$ = 0 for r=a $\Rightarrow D = -Ua^2$

No choice of D will satisfy both these conditions. Alternatively, had we satisfied the near-field boundary conditions first it would have been found that it is impossible to satisfy the far-field conditions. Thus, we conclude that there is no solution to Stokes equations in 2D which can satisfy both the near- and far-field conditions: Stokes paradox.

Recall that solutions to Stokes equations can be expressed in terms of asymptotic or perturbation expansions

$$\psi = \psi^0 + Re\psi^1 + Re^2\psi^2 + \cdots$$

We see that for 2D the O(0) expansion is singular (i.e., has no solution). For 3D, it can be shown that the O(Re) expansion is singular (Whitehead paradox). Such expansions are called singular perturbation expansions or nonuniform expansions.

The reason for the singular behavior is the fact that inertia is not negligible in the far field. In the far field, the inertia force is actually larger than the viscous force since convection velocity is ~ U and velocity gradients are small.

An alternative low Re approximation put forth by Oseen to remove this difficulty is

$(\underline{V} \cdot \nabla)\underline{V} \approx U \frac{\partial \underline{V}}{\partial x}$ linearized inertia term

Oseen equations:

$$\left.\begin{aligned} \nabla \cdot \underline{V} &= 0 \\ \rho\left[\frac{\partial \underline{V}}{\partial t} + \mathrm{U} \cdot \frac{\partial \underline{V}}{\partial x}\right] &= -\nabla p + \mu \nabla^2 \underline{V} \end{aligned}\right\} \text{ still linear!}$$

Solutions to Oseen equations can be obtained in a similar manner as for Stokes equations; however, these can be shown to not be accurate in the near field. The remedy is to use the technique of matched asymptotic expansions whereby the near and far solutions are matched to provide a uniformly valid approximation. (see Van Dyke: Perturbation Methods in Fluid Mechanics)

Some solutions to the Oseen equations are given in the text for a sphere, flat plate normal to the flow, and a circular cylinder. Note that the velocity field predicted for a sphere no longer shows fore-aft symmetry but now includes a broader aft wake. However, there is no improvement in the prediction of C_D.