Chapter 3 Solutions of the Newtonian Viscous-Flow Equations

8. Similarity solutions

Similarity solutions have the following properties:

(1) reduction of variables by one or more through analytical means (coordinate transformation)

(2) reduction of pde to ode's

(3) physically: inertia, pressure, and viscous forces are in some constant proportion (e.g., Re based on local scales is constant)

a. Stagnation point

b. Rotating disc

c. Convergent/divergent channels

Also, others regarding the BL equations and turbulent flow

Stagnation Point Flow

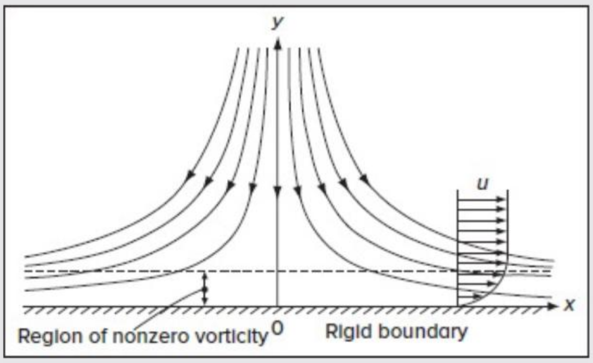


FIGURE 3-31

Stagnation flow in both planar and axisymmetric configurations. In the latter, it is customary to replace *x* by the radial coordinate *r*.

Inviscid Flow

$$\psi = Bxy \qquad u = Bx = \psi_y \qquad v = -By = -\psi_x \qquad B = \frac{U_0}{L}$$

$$v = 0 \text{ along wall} \quad u = 0 \text{ only at stagnation point}$$

$$p + \frac{1}{2}\rho(u^2 + v^2) = \text{constant}$$

$$p_0 \; = \; stagnation\; pressure \;\; p = p_0 - \tfrac{1}{2}\rho B^2(x^2 + y^2)$$

For x = 0, consider $U(y) = U_0\left(1 - \frac{y}{L}\right)$

$$p + \frac{1}{2}\rho U^2 = C$$

$$p_y + \rho U U_y = 0$$

$$p_y = -\rho U U_y \; = \rho\frac{U_0^2}{L}\left(1 - \frac{y}{L}\right)$$ Adverse pressure gradient

Howarth linearly decelerating flow used 2D integral method BL with adverse pressure gradient.

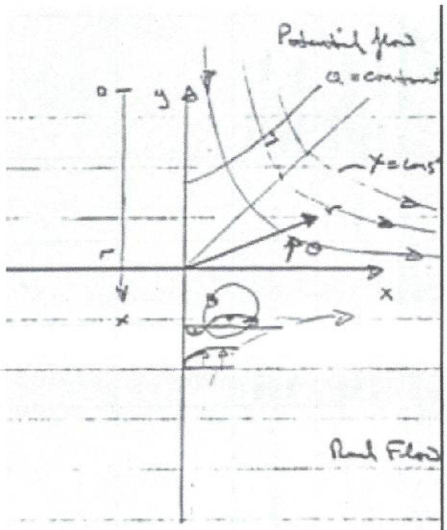
Consider the complex potential

$$F(z) = \frac{a}{2}z^2 = \frac{a}{2}r^2 e^{2i\theta}$$

$$\varphi = \mathrm{Re}[F(z)] = \frac{a}{2}r^2 \cos 2\theta$$

$$\psi = \mathrm{Im}[F(z)] = \frac{a}{2}r^2 \sin 2\theta$$

Orthogonal rectangular hyperbolas



$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

φ: asymptotes y = ± x

ψ : asymptotes x=0, y=0

$$\begin{cases} \underline{V} = \nabla\varphi = \varphi_r \hat{e}_r + \frac{1}{r}\varphi_\theta \hat{e}_\theta \\ v_r = ar\cos 2\theta \\ v_\theta = -ar\sin 2\theta \end{cases}$$

$\frac{\pi}{2} \le \theta \le 0$ (flow direction as shown)

$$\underline{V} = v_r(\cos\theta\hat{i} + \sin\theta\hat{j}) + v_\theta(-\sin\theta\hat{i} + \cos\theta\hat{j}) =$$
$$(v_r\cos\theta - v_\theta\sin\theta)\hat{i} + (v_r\sin\theta + v_\theta\cos\theta)\hat{j}$$

Potential flow slips along surface: (consider $\theta = 90^\circ$)

1) determine a such that $v_r = U_0$ at r=L, $\theta = 90^\circ$

   $v_r = aL\cos(2\times 90) = U_0 \Rightarrow aL = -U_0$, i.e. $a = -\frac{U_0}{L}$

2) let $U(x) = v_r$ at x=L-r:

   $\Rightarrow v_r = a(L-x)\cos(2\times 90) = U(x)$

   $Or:\ U(x) = -a(L-x) = \frac{U_0}{L}(L-x) = U_0(1-\frac{x}{L})$

$\psi = Bxy \quad B = U/L$

$u = Bx = \psi_y$

$v = -By = -\psi_x$

$p + \frac{1}{2}\rho(u^2+v^2) = C$

$p + \frac{1}{2}\rho B^2(x^2+y^2) = C$

$p(0,0) = C = p_0$

$p = p_0 - \frac{1}{2}\rho\frac{U^2}{L^2}(x^2+y^2)$

$p_x = -\rho\frac{U^2}{L^2}x$

$p_y = -\rho\frac{U^2}{L^2}y$

$U_x = -\frac{U_0}{L}$

$p + \frac{1}{2}\rho U^2 = C$

$p_x + \rho U U_x = 0$

$p_x = -\rho U U_x = -\rho U_0(1-\frac{x}{L})(-\frac{U_0}{L}) = \rho\frac{U_0^2}{L}(1-\frac{x}{L})$

Viscous Flow

Analysis can be extended curved surface if flow region small, compared radius of curvature

$$u_x + v_y = 0 \qquad (1)$$

$$uu_x + vu_y = -\frac{p_x}{\rho} + \nu(u_{xx} + u_{yy}) \qquad (2)$$

$$uv_x + vv_y = -\frac{p_y}{\rho} + \nu(v_{xx} + v_{yy}) \qquad (3)$$

$$\psi \quad = \quad Bx \underbrace{f(y)}_{\neq y \text{ as in } \mu=0 \text{ solution}}$$

$$u = Bxf'$$

$$v = -Bf$$

No slip: $f(0) = f'(0) = 0$

From continuity: $v_y = -u_x = -Bf'$

Substitution u and v into (2) and (3)

$$B^2xf'^2 - B^2xff'' = -\frac{p_x}{\rho} + \nu Bxf''' \qquad (4)$$

$$B^2ff' = -\frac{p_y}{\rho} - \nu Bf'' \qquad (5)$$

from (5) $p_y = -\rho B(Bff' + \nu f'') = f(y) \therefore p_{yx} = 0$

$$p(x,y) = -\rho B\left(\frac{B}{2}f^2 + \nu f'\right) + g(x)$$

Determine $g(x)$ from condition that for large y recover potential flow

i.e. $f(\infty) = y, \qquad f'(\infty) = 1, \qquad f''(\infty) = f'''(\infty) = 0$

$$p(x,\infty) = -\rho B\left(\frac{B}{2}y^2 + \nu\right) + g(x) = p_0 - \frac{1}{2}\rho B^2(x^2 + y^2)$$

$$g(x) = p_0 - \frac{1}{2}\rho B^2x^2 + \rho B\,\nu$$

$$p(x,y) = p_0 - \frac{1}{2}\rho B^2f^2 + \rho B\nu(1 - f') - \frac{1}{2}\rho B^2x^2 \qquad (6)$$

$p_x = -\rho B^2x \;\; p_y = -\frac{1}{2}\rho B^2 2ff' + \rho B\nu f'' = -\rho B(Bff' + \nu f'')$

Substitution into (4)

$$B^2xf'^2 - B^2xff'' = B^2x + \nu Bxf'''$$

or $\frac{\nu}{B}f''' + ff'' - f'^2 + 1 = 0 \qquad (7)$

Alternative derivation:

from (3)

$$\frac{p_y}{\rho} = -\cancel{uv_x} - vv_y + \nu\left(\cancel{v_{xx}} + v_{yy}\right)$$

$$= Bf(-Bf') - \nu Bf''$$

$$p_y = -\rho B(Bff' + \nu f'') = f(y) \quad \therefore\ p_{yx} = 0$$

$$p = -\rho B\left(\nu f' + \frac{B}{2}f^2\right) + \varphi_1(x) + C \qquad (4)$$

from (2)

$$\frac{p_x}{\rho} = -uu_x - vu_y + \nu\left(\cancel{u_{xx}} + u_{yy}\right)$$

$$= -Bxf'(Bf') + Bf(Bxf'') + \nu(Bxf''')$$

$$p_x = \rho Bx(\nu f''' + Bff'' - Bf'^2) \qquad (5)$$

$$p = \frac{1}{2}\rho Bx^2(\nu f''' + Bff'' - Bf'^2) + \varphi_2(y) + C \qquad (6)$$

Comparing (4) and (6) or considering $\frac{\partial(5)}{\partial y} = 0$

$$f''' + \frac{B}{\nu}(ff'' - f'^2) = \text{constant} = -\frac{B}{\nu}$$

For large y, $u = Bx \quad f'(\infty) = 1, \qquad f''(\infty) = 0, \qquad f'''(\infty) = 0$

$$f''' + \frac{B}{\nu}(ff'' - f'^2 + 1) = 0 \qquad (7)$$

non dimensionalize using length scale $\sqrt{\nu/B}$ and velocity scale $\sqrt{\nu B}$, recall ν m^2/s and $B = U_0/L \cdot 1/s$

Alternate derivation using bi-harmonic ψ equation:

$$\psi_y \frac{\partial}{\partial x}(\nabla^2\psi) - \psi_x \frac{\partial}{\partial y}(\nabla^2\psi) = \nu\nabla^4 \psi$$

$$\omega_z = v_x - u_y = -\nabla^2\psi = -(\psi_{xx} + \psi_{yy}) = v_x - u_y$$

$$u = \psi_y \qquad v = -\psi_x$$

$$\nabla^4\psi = \psi_{xxxx} + \psi_{yyyy} = -v_{xxx} + u_{yyy} \qquad \nabla^2\psi = u_y - v_x$$

$$u\frac{\partial}{\partial x}(u_y - v_x) + v\frac{\partial}{\partial y}(u_y - v_x) = \nu(-v_{xxx} + u_{yyy})$$

$$u(u_{yx} - v_{xx}) + v(u_{yy} - v_{xy}) = \nu(-v_{xxx} + u_{yyy})$$

Stagnation Flow: $\psi = Bxf, \qquad u = Bxf', \qquad v = -Bf$

$$Bxf'(Bf'') - Bf(Bxf''') = \nu(Bxf^{(iv)})$$

$$\nu Bxf^{(iv)} + B^2xff''' - B^2xf'f'' = 0$$

$$\frac{\nu}{B}f^{(iv)} + ff''' - f'f'' = 0$$

integrate: $\frac{\nu}{B}f''' + ff'' - f'^2 = \text{constant}$

$f''' + \frac{B}{\nu}(ff'' - f'^2) = -\frac{B}{\nu}$ since for large y, $u = Bx \Rightarrow$
$f'(\infty) = 1, \quad f''(\infty) = f'''(\infty) = 0$

finally: $f''' + \frac{B}{\nu}(ff'' - f'^2 + 1) = 0$

$$\frac{\nu}{B} f''' + ff'' - f'^2 + 1 = 0 \qquad (7)$$

Boundary conditions:

$u(x, 0) = 0 \;\Rightarrow\; f'(0) = 0$

$v(x, 0) = 0 \;\Rightarrow\; f(0) = 0$

and for matching with potential flow

$$f'(\infty) = 1$$

non dimensionalize using length scale $\sqrt{\nu/B}$ and velocity scale $\sqrt{\nu B}$ recall $\nu = m^2/s \quad B = \frac{U_0}{L} \quad \frac{1}{s}$

$y = \sqrt{B/\nu}\,\eta$

$\frac{\partial}{\partial y} = \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \sqrt{B/\nu}\,\frac{\partial}{\partial \eta}$

$F = \sqrt{B/\nu}\, f$

$f = \sqrt{\nu/B}\; F$

$$\eta = y\sqrt{B/\nu}, \qquad \psi = xF(\eta)\sqrt{\nu B} \qquad \psi = \frac{m^2}{s}$$

$$u = BxF', \qquad v = -F\sqrt{\nu B}$$

(7) becomes: $F''' + FF'' + 1 - (F')^2 = 0$ nonlinear, 3rd order ODE

$$F(0) = 0, \qquad F'(0) = 0, \qquad F'(\infty) = 1$$ solved numerically

Note: $f' = \sqrt{\nu/B}\; F'\sqrt{B/\nu} = F'$

$f'' = \sqrt{B/\nu}\; F''$

$f''' = B/\nu\; F'''$

Asymptotic analysis: for $F''' + FF'' + 1 - F'^2 = 0$, as η becomes large, $F \to a + \eta$ and $(1 - F'^2) \to 0$. Therefore, for large η:

$$F'''/F'' = \text{-F} = -\eta \text{ i.e. } F'' \approx e\, -\eta^2/2$$

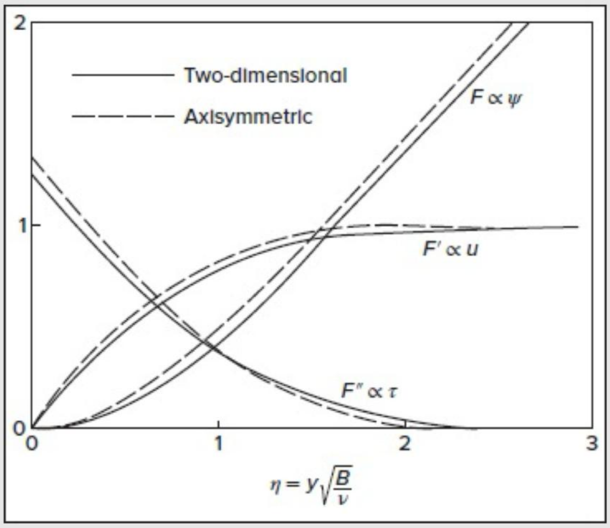


FIGURE 3-32
Numerical solutions of viscous stagnation flow for planar [Eq. (3-206)] and axisymmetric [Eq. (3-219)] conditions.

$F' = u/U$		
η	**Planar** $F''(0) = 1.23259$ $\eta^* = 0.6479$	**Axisymmetric** $F''(0) = 1.31194$ $\eta^* = 0.5689$
0.1	0.11826	0.12619
0.2	0.22661	0.24239
0.3	0.32524	0.34863
0.4	0.41446	0.44499
0.5	0.49465	0.53160
0.6	0.56628	0.60871
0.7	0.62986	0.67663
0.8	0.68954	0.73577
0.9	0.73508	0.78666
1.0	0.77787	0.82987
1.1	0.81487	0.86608
1.2	0.84667	0.89598
1.3	0.87381	0.92032
1.4	0.89681	0.93983
1.5	0.91617	0.95522
1.6	0.93235	0.96718
1.7	0.94578	0.97631
1.8	0.95684	0.98316
1.9	0.96588	0.98822
2.0	0.97322	0.99190
2.2	0.98386	0.99635
2.4	0.99055	0.99847
2.6	0.99464	0.99940
2.8	0.99705	0.99979
3.0	0.99843	0.99993

Solution shows BL behavior.

1. Viscous flow merges smoothly with outer flow: U(x) = u(x,∞) = Bx.

2. δ = thickness stagnation layer. For u/U = 0.99

$$\eta = 2.4 = \frac{y}{\sqrt{B/\nu}} = \frac{\delta}{\sqrt{B/\nu}} \Rightarrow \delta = 2.4\sqrt{\nu/B} \neq f(x)$$

The boundary layer thickness is constant because thinning due to favorable axial pressure gradient exactly balances the thickening caused by viscous diffusion.

δ small: air 20°C approaching 10 cm D cylinder at $U_0 = 10\ m/s$

$B = \frac{U_0}{L} = \frac{4U_0}{D} = 400\ \mathrm{s}^{-1}$and $\delta \approx 0.46\ \mathrm{mm}$ about .5%D.

3. Displacement thickness (Appendix A)

Another boundary-layer effect is the displacement of the outer stream by the shear layer, as hinted at in Fig. 3-31. We define the *displacement thickness* δ^* as the distance the outer inviscid flow is pushed away from the wall because of the decelerating viscous layer. In terms of the stream function F, we find that

$$\lim_{\eta\to\infty} F(\eta) = \eta - \eta^* \qquad \text{where} \qquad \eta^* = \delta^*\sqrt{\frac{B}{\nu}} = 0.6479 \qquad (3\text{-}211)$$

as shown in Table 3-4. In stagnation flow, then, $\delta^* \approx 0.27\delta$.

4. Pressure distribution has same behavior BL

$p_x = -\rho B^2 = -\rho U U_x$ inviscid flow pressure gradient

$$p_y = -\rho B(Bff' + \nu f'')$$

$$= -\rho B\left(B\sqrt{\nu/B}\ FF' + \nu\sqrt{B/\nu}\ F''\right)$$

$$= -\rho B\sqrt{\nu B}\,(FF' + F'') = O(\sqrt{\nu})$$

Similar as boundary layer theory, i.e. $p_y \approx 0$ across δ

5. Shear stress

$$\tau_w = \mu(u_y + v_x)|_{y=0}$$

$$\tau_w = \mu Bxf'' = \mu Bx\sqrt{B/\nu}\; F''|_{y=0}$$

$$\tau_w = U\sqrt{\mu\rho B}\, F_0'' \;\propto\; U$$

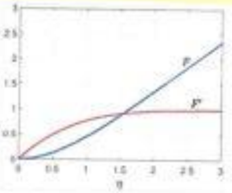
$$C_f = \frac{\tau_w}{1/2\rho U^2} = \frac{2F_0''}{\sqrt{Re_x}} \propto \mathrm{Re}_x^{-1/2} \qquad Re_x = \frac{Ux}{\nu}$$

Again, similar laminar boundary layer theory.

6. Also exact solution BL equations, i.e., Falkner-Skan similarity solution $\alpha = \beta = 1$

Many extensions of theory such as: oblique flow, two fluid, axisymmetric, circular cylinder, & unsteady flows

The flow due to the presence of a solid surface at $z = 0$ in planar stagnation-point flow was described first by Karl Hiemenz in 1911,[6] whose numerical computations for the solutions were improved later by Leslie Howarth.[7] A familiar example where Hiemenz flow is applicable is the forward stagnation line that occurs in the flow over a circular cylinder.[8][9]



Two-dimensional stagnation point flow

The solid surface lies on the xy. According to potential flow theory, the fluid motion described in terms of the stream function ψ and the velocity components $(v_x, 0, v_z)$ are given by

$$\psi = kxz, \quad v_x = kx, \quad v_z = -kz.$$

The stagnation line for this flow is $(x, y, z) = (0, y, 0)$. The velocity component v_x is non-zero on the solid surface indicating that the above velocity field do not satisfy no-slip boundary condition on the wall. To find the velocity components that satisfy the no-slip boundary condition, one assumes the following form

$$\psi = \sqrt{\nu k}xF(\eta), \quad \eta = \frac{z}{\sqrt{\nu/k}}$$

where ν is the Kinematic viscosity and $\sqrt{\nu/k}$ is the characteristic thickness where viscous effects are significant. The existence of constant value for the viscous effects thickness is due to the competing balance between the fluid convection that is directed towards the solid surface and viscous diffusion that is directed away from the surface. Thus the vorticity produced at the solid surface is able to diffuse only to distances of order $\sqrt{\nu/k}$; analogous situations that resembles this behavior occurs in asymptotic suction profile and von Kármán swirling flow. The velocity components, pressure and Navier–Stokes equations then become

$$v_x = kxF', \quad v_z = -\sqrt{\nu k}F, \quad \frac{p_o - p}{\rho} = \frac{1}{2}k^2x^2 + k\nu F' + \frac{1}{2}k\nu F^2$$

$$F''' + FF'' - F'^2 + 1 = 0$$

The requirements that $(v_x, v_z) = (0, 0)$ at $z = 0$ and that $v_x \to kx$ as $z \to \infty$ translate to

$$F(0) = 0, \ F'(0) = 0, F'(\infty) = 1.$$

The condition for v_z as $z \to \infty$ cannot be prescribed and is obtained as a part of the solution. The problem formulated here is a special case of Falkner-Skan boundary layer. The solution can be obtained from numerical integrations and is shown in the figure. The asymptotic behaviors for large $\eta \to \infty$ are

$$F \sim \eta - 0.6479, \quad v_x \sim kx, \quad v_z \sim -k(z - \delta^*), \quad \delta^* = 0.6479\delta$$

where δ^* is the displacement thickness.

Another exact solution to the boundary-layer equations that may be obtained from the Falkner-Skan similarity solution is that corresponding to a stagnation-point flow. The values of the constants α and β that yield this solution are $\alpha = \beta = 1$. But this is equivalent to letting β be unity in the solution for the flow over a wedge. Then the angle of the wedge becomes π, which means the flow impinges on a flat surface yielding a plane stagnation point.

The solution may be obtained by setting $\beta = 1$ in Eqs. (9.7). This gives

$$U(x) = cx \tag{9.8a}$$

$$\xi(x) = \sqrt{\frac{\nu}{c}} \tag{9.8b}$$

$$f''' + ff'' + 1 - (f')^2 = 0 \tag{9.8c}$$

$$f(0) = f'(0) = 0 \tag{9.8d}$$

$$f'(\eta) \to 1 \quad \text{as } \eta \to \infty \tag{9.8e}$$

$$\psi(x, y) = \sqrt{c\nu}\, xf\left(\frac{y}{\sqrt{\nu/c}}\right) \tag{9.8f}$$

It will be noticed that this is precisely the exact solution to the full Navier-Stokes equations that was obtained by Hiemenz for a stagnation point. This solution is given by Eqs. (7.7*a*), (7.7*b*), and (7.7*c*). Thus the exact solution to the boundary-layer equations is also an exact solution to the full Navier-Stokes equations in this instance.

3.81 Flow near an infinite rotating disk

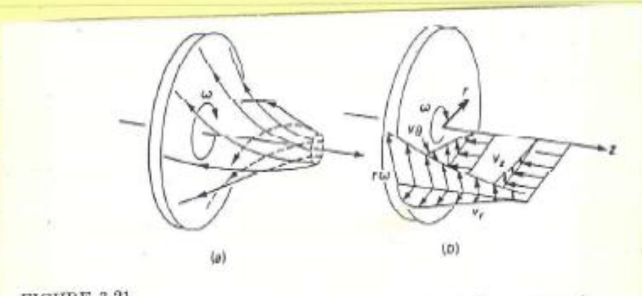


FIGURE 3-21
Laminar flow near a rotating disk: (a) streamlines; (b) velocity components.

Flow is rotationally symmetric : $\frac{\partial}{\partial \theta} = 0$

(1) $\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial}{\partial z}(v_z) = 0$

(2) $v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2}\right)$

(3) $v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z} - \frac{1}{r} v_r v_\theta = \nu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2}\right)$

(4) $v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right)$

The boundary conditions are :

$z = 0$: $v_r = v_z = 0$ $\quad v_\theta = r\omega$ $\quad p = 0$ ← convenient constant

$z \to \infty$: $v_r = v_\theta = 0$

note : $v_z(\infty)$ not prescribed (centrifugal pumping)
also $\frac{\partial p}{\partial r} = 0$ so that p is uncoupled
from $\underline{v}$ + can be solved from (1) – (3) & then (4) for p

There are no obvious velocity and length scales for this problem.

Karman (1921): $\frac{v_r}{r}$, $\frac{v_\theta}{r}$, v_z, $p = f(z)$ only

$\Rightarrow$ define $z^* = z/\sqrt{\nu/\omega}$ = similarity variable (Similar to $y/\sqrt{\nu t}$ + others)

$$v_r = r\omega F(z^*) \quad v_\theta = r\omega G(z^*) \quad v_z = \sqrt{\omega\nu}\, H(z^*)$$

$$p = \rho\omega\nu P(z^*)$$

Substitution into (1)-(4) results in

$$\underbrace{H' = -2F \quad F'' = -G + F^2 + F'H \quad G'' = 2FG + HG'}_{\text{Solve for } F, G, H}$$

$$\underbrace{P' = 2FH - 2F'}_{\text{then solve for } p}$$

$'$ denotes $\frac{\partial}{\partial z^*}$

$$F(0) = H(0) = P(0) = 0 \quad G(0) = 1$$
$$F(\infty) = G(\infty) = 0$$

Solution is obtained by using standard techniques for system of ODE (see discussion in text)

Reality: laminar flow unstable at $Re = 3.2\times10^5$

$Re = \frac{\omega r^2}{\nu}$

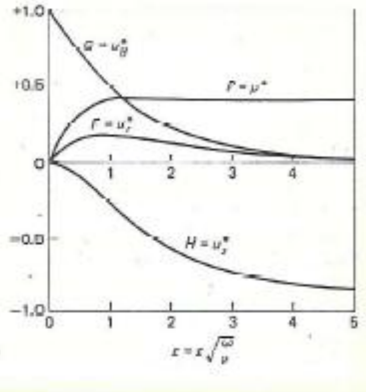


FIGURE 3-23
Numerical solutions of Eqs. (3-155) for the infinite rotating disk.

$\delta \simeq 5\sqrt{\nu/\omega}$ $\quad \delta/r_0 \approx 5/\sqrt{Re}$ (actually independent of r) $\quad Re = \frac{\omega r_0^2}{\nu}$

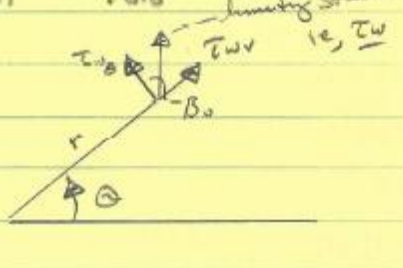
$Q = 2\pi r \int_0^\infty (\ldots)\,dz = .885\,\pi r^2 \sqrt{\nu\omega}$ = amount of fluid pumped outward on one side of the disk

$\Rightarrow v_z(\infty) = .885\sqrt{\nu\omega}$ (volume flow per unit area)

$\underline{\tau_w} = \tau_{wr}\,\hat{e}_r + \tau_{w\theta}\,\hat{e}_\theta$

$\tau_{wr} = \mu \frac{\partial v_r}{\partial z}\Big|_w \qquad \tau_{w\theta} = \mu \frac{\partial v_\theta}{\partial z}\Big|_w$

$\tan\beta = -\frac{\tau_{wr}}{\tau_{w\theta}} = -\frac{F'(0)}{G'(0)} = \frac{.51}{.616} = .828$



$\beta = 39.6°$

3.82 Moment Coefficient

Above solution is strictly only applicable to an infinite disk. However, it should also be valid for a finite disk if

$r_0 \gg \delta \qquad R_e \gg 1$

then departures are limited to thin edge effects. It is of interest to calculate the torque on the disk

$$M = \int \tau_{w_\theta} \, r \, dA = \int_0^{r_0} \tau_{w_\theta} \, 2\pi r^2 \, dr$$

$\tau_{w_\theta} = \rho \nu \sqrt{\nu \omega^3} \, G_0'$

$$= \frac{\pi}{2} \rho r_0^4 G_0' (\nu \omega^3)^{1/2}$$

$G_0' = -.616$

$$C_m = \frac{-2M}{\frac{1}{2} \rho \omega^2 r_0^5} = \frac{3.87}{\sqrt{R_e}} \qquad R_e = \frac{\omega r_0^2}{\nu}$$

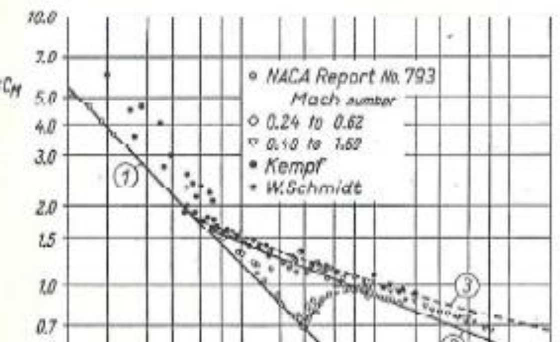


Fig. 5.14. Turning moment on a rotating disk; curve (1) from eqn. (5.56), *laminar*; curves (2) and (3) from eqns. (21.30) and (21.33), *turbulent*

good agreement ← | → flow becomes turbulent

Flow in convergent / divergent channel

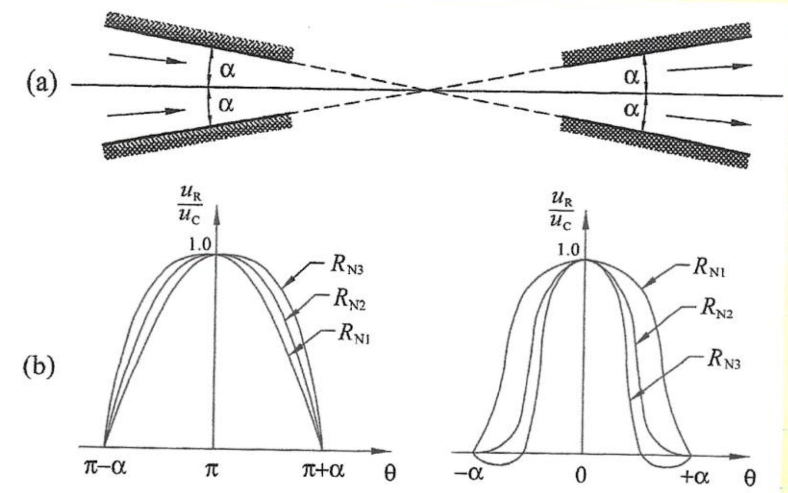


FIGURE 7.7 (a) Flow configuration, and (b) velocity profiles for flow in convergent and in divergent channels.

$R_{N1} > R_{N2} > R_{N3}$
Convergent
different
divergent
especially
low R_N

Adverse
pressure
gradient
divergent
induces
back flow
near walls
(low inertia)
especially
large α

Cylindrical coordinates (R, θ, z) with only $u_R = f(R, \theta)$

$$\frac{1}{R}\frac{\partial}{\partial R}(R u_R) = 0$$

$$u_R \frac{\partial u_R}{\partial R} = -\frac{1}{\rho} P_R + \nu \left[R^{-1}\frac{\partial}{\partial R}\left(R \frac{\partial u_R}{\partial R}\right) - \frac{u_R}{R^2} + \frac{1}{R^2}\frac{\partial^2 u_R}{\partial \theta^2} \right]$$

$$R\frac{\partial^2 u_R}{\partial R^2} + \frac{\partial u_R}{\partial R}$$

$$\left[\frac{\partial^2 u_R}{\partial R^2} + R^{-1}\frac{\partial u_R}{\partial R} - \frac{u_R}{R^2} + \frac{1}{R^2}\frac{\partial^2 u_R}{\partial \theta^2} \right]$$

$$0 = -\frac{1}{\rho R} P_\theta + \nu \left(\frac{2}{R^2}\frac{\partial u_R}{\partial \theta} \right)$$

Separation of variables: $u_R(R, \theta) = f(R) F(\theta)$

Continuity ⇒ $R u_R$ = constant so $u_R(R,\theta) = \frac{\nu}{R} F(\theta)$

ie $u_R \propto R^{-1}$ ν makes $F(\theta)$ dimensionless

$$\frac{\partial u_R}{\partial R} = -\frac{\nu}{R^2} F(\theta) \qquad \frac{\partial u_R}{\partial \theta} = \frac{\nu}{R} F'$$

$$\frac{\partial^2 u_R}{\partial R^2} = +\frac{2\nu}{R^3} F(\theta) \qquad \frac{\partial^2 u_R}{\partial \theta^2} = \frac{\nu}{R} F''$$

$$-\frac{\nu^2}{R^3} F^2 = -\rho^{-1} p_R + \nu \left[+\frac{2\nu}{R^3} F - \frac{\nu}{R^3} F - \frac{\nu}{R^3} F + \frac{\nu}{R^3} F'' \right]$$

$$-\frac{\nu^2}{R^3} F^2 = -\rho^{-1} p_R + \frac{\nu^2}{R^3} F''$$

$$0 = -\frac{1}{\rho R} p_\theta + \frac{2\nu}{R^2} \left(\frac{\nu}{R} F' \right) = -\frac{1}{\rho R} p_\theta + \frac{2\nu^2}{R^3} F'$$

$$-\frac{\nu^2}{R^3} (2 F F') = -\frac{1}{\rho} p_{R\theta} + \frac{\nu^2}{R^3} F'''$$

$$0 = -\frac{1}{\rho} p_\theta + \frac{2\nu^2}{R^2} F'$$

$$0 = -\frac{1}{\rho} p_{\theta R} - \frac{4\nu^2}{R^3} F'$$

$$-\frac{\nu^2}{R^3} (2 F F') = \frac{\nu^2}{R^3} F''' + \frac{4\nu^2}{R^3} F'$$

$$-2FF' = F''' + 4F' \quad \text{or} \quad F''' + 4F' + 2FF' = 0$$

integrate wrt θ $\Rightarrow$ $F'' + 4F + (F^2) = K$ = constant

let $G(F) = F'$ G = new dependent and F independent variables

$$\frac{dG}{dF} = F'' = \frac{d\theta}{dF} \frac{d}{d\theta} (F') = \frac{F''}{F'} = \frac{F''}{G} \Rightarrow F'' = G \frac{dG}{dF}$$

$\therefore$ $G \dfrac{dG}{dF} + 4F' + F^2 = K$ 1st order ODE in G

$$\frac{d}{dF}\left(\tfrac{1}{2} G^2\right) = K - 4F' - F^2$$

integrate wrt F

$$\frac{1}{2}G^2 = A + KF - 2F^2 - \frac{1}{3}F^3$$

$$G(F) = \frac{dF}{d\theta} = \left[2\left(A + KF - 2F^2 - \frac{1}{3}F^3\right)\right]^{1/2}$$

$$\theta = \int_0^F \frac{dF}{\left[2\left(A + KF - 2F^2 - \frac{1}{3}F^3\right)\right]^{1/2}} + B$$

where $u_R(R,\theta) = \frac{\nu}{R} F(\theta)$

Solve numerically with BC

$u_R(\alpha) = u_R(-\alpha) = 0$ divergent
$u_R(\pi+\alpha) = u_R(\pi-\alpha) = 0$ convergent — no slip

$\frac{\partial u_R}{\partial \theta}(R,0) = 0$ divergent
$\frac{\partial u_R}{\partial \theta}(R,\pi) = 0$ convergent — symmetry

in terms of $F(\theta)$: $F(\alpha) = F(-\alpha) = F'(0) = 0$ divergent
$F(\pi+\alpha) = F(\pi-\alpha) = F'(\pi) = 0$ convergent

which determines A, B, K

Solutions in terms of $R_D = \frac{u_c R}{\nu}$ — u_c = centerline velocity