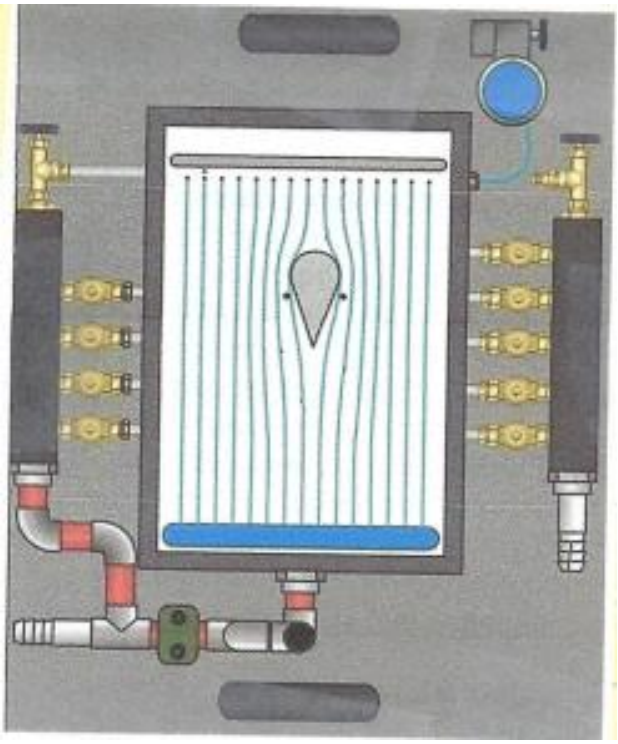
Hele–Shaw Flow



Table/cell = thin flat chamber, constant h, flow driven by p or g.

Originally for study lubrication by Hele–Shaw, inventor fluid clutch & variable pitch propeller.

2D viscous dominated flow with potential flow streamlines. Wall shapes and obstacles can be used to visualize ideal flow patterns including singularities i.e. point sources/sinks.

Consider lubrication approximation with (x, y) main flow scaled L, $z =$ width scaled h, and $\varepsilon = h/L \to 0$

$$0 = -p_x + \mu u_{zz} \qquad 0 = -p_y + \mu v_{zz} \qquad 0 = -p_z$$

$$z = 0 \qquad (u, v, w) = 0$$
$$z = h \qquad (u, v, w) = 0$$

$$u = \frac{h^2}{2\mu} p_x \left[\left(\frac{z}{h}\right)^2 - \frac{z}{h}\right]$$

$$v = \frac{h^2}{2\mu} p_y \left[\left(\frac{z}{h}\right)^2 - \frac{z}{h}\right]$$

$$w = 0$$

$$p = p(x, y)$$

$\psi \neq \psi(z)$ and $\left.\frac{dx}{dy}\right|_{\psi} = \frac{u}{v} = \frac{p_x}{p_y} = f(x,y)$

and in z direction ψ are uniformly on top of each other.

$\omega_z = -u_y + v_x = \left[-p_{xy} + p_{yx}\right] \frac{h^2}{2\mu}\left[\left(\frac{z}{h}\right)^2 - \frac{z}{h}\right] = 0$, i.e., $\nabla \times \underline{V} = 0$

such that $\underline{V} = \nabla\varphi$ and from continuity $u_x + v_y = 0$, i.e., $\nabla \cdot \underline{V} = 0$, which shows $\nabla^2\varphi = 0 =$ ideal flow.

However, p not governed by Bernoulli equation.

$$u_x = ap_{xx}, \quad v_y = ap_{yy}, \quad a = \frac{h^2}{2\mu}\left[\left(\frac{z}{h}\right)^2 - \frac{z}{h}\right]$$

$\therefore p_{xx} + p_{yy} = 0$ as per Stokes flow

$\underline{V} = u\hat{\imath} + v\hat{\jmath}$ with $u = ap_x$ and $v = ap_y$

$$\nabla p = p_x\hat{\imath} + p_y\hat{\jmath} = a(u\hat{\imath} + v\hat{\jmath})$$

i.e. $\nabla p \propto \underline{V}$ and aligned $\underline{V}$ such that $\underline{V} \times \nabla p = 0$

$$(u\hat{\imath} + v\hat{\jmath}) \times (p_x\hat{\imath} + p_y\hat{\jmath}) = 0 \qquad up_y\hat{k} - vp_x\hat{k} = 0$$

$$ap_xp_y - ap_yp_x = 0 = a(p_xp_y - p_yp_x)$$

$(u,v)_{max}$ at $z = h/2$ and from velocity profiles

$$p_x = -\frac{8\mu}{h^2}u_0 \qquad p_y = -\frac{8\mu}{h^2}v_0 \qquad \underline{V_0} = u_0\hat{\imath} + v_0\hat{\jmath}$$

$$\Delta p = \int (p_x\,dx + p_y\,dy) = -\frac{8\mu}{h^2}\left[\int u_0\ dx + \int v_0\ dy\right]$$

$$p(x,y) - p_0 = -\frac{8\mu}{h^2}\int_{\psi_{\text{midplane}}} V_0\ ds$$

$$\widehat{e_s} = dx\,\hat{\imath} + dy\,\hat{\jmath} = \underline{ds}$$

$$\underline{V_0} = u_0\hat{\imath} + v_0\hat{\jmath} = V_0\ \underline{ds}$$

$$\underline{V_0} \cdot \underline{ds} = u_0\,dx + v_0\,dy$$

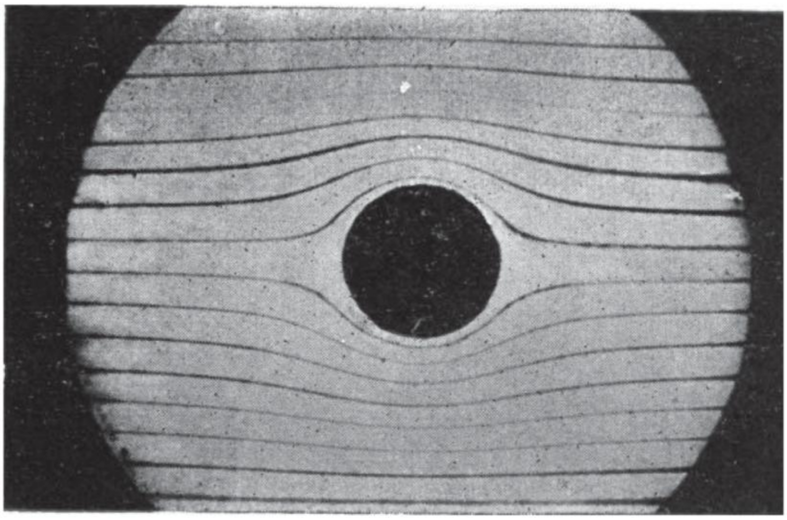


Figure 22.8 Flow around circular cylinder model in a Hele-Shaw apparatus. Streamlines form the potential flow pattern. Photograph courtesy of Professor Bloor, University of Edinburgh, is attributed to Institute of Naval Architects in Bloor (2008).

Example: circular cylinder

potential flow solution $\underline{V} = u_r \widehat{e_r} + u_\theta \widehat{e_\theta}$

$u_r = U(1 - r_0^2/r^2)\cos\theta \quad u_\theta = -U(1 + r_0^2/r^2)\sin\theta$

U = midplane velocity far from cylinder along stagnation streamline which intersects body at $r = R$, $\theta = \pi$ where $V_0 = u_r$

↑ far from cylinder

$$p(r, \pi) - p(R, \pi) = -\frac{8\mu U}{h^2}\left(R + \frac{r_0^2}{R} - r - \frac{r_0^2}{r}\right)$$

Then from $r = r_0$ to $\theta = \pi \quad V_0 = u_\theta$

$$p(r_0, \theta) - p(r_0, \pi) = -\frac{16\mu U}{h^2}(1 + \cos\theta)$$

Along the stagnation point ψ and around the cylinder the pressure decreases

$$p(x,y) - p_0 = -\frac{8\mu}{h^2}\int_{\psi} V_0 \; ds$$

$$p(r,\pi) - p(R,\pi) = -\frac{8\mu}{h^2}\int_{r=R}^{r} U\left(1 - \frac{r_0^2}{r^2}\right)\cos\theta \; dr$$

$$= \frac{8\mu U}{h^2}\int_R^r \left(1 - \frac{r_0^2}{r^2}\right) dr \; = \frac{8\mu U}{h^2}\left[r + \frac{r_0^2}{r}\right]_R^r = -\frac{8\mu U}{h^2}\left[R + \frac{r_0^2}{R} - r - \frac{r_0^2}{r}\right]$$

$$p(r_0,\theta) - p(r_0,\pi) = -\frac{8\mu}{h^2}\int_{\theta}^{\theta=\pi} U\left(1 + \frac{r_0^2}{r_0^2}\right)\sin\theta \; d\theta$$

$$= -\frac{16\mu U}{h^2}\left[-\cos\theta\right]_\theta^\pi \; = -\frac{16\mu U}{h^2}(1 + \cos\theta)$$

The **Saffman–Taylor instability**, also known as **viscous fingering**, is the formation of patterns in a morphologically unstable interface between two fluids in a porous medium, described mathematically by Philip Saffman and G. I. Taylor in a paper of 1958.[1][2] This situation is most often encountered during drainage processes through media such as soils.[3] It occurs when a less viscous fluid is injected, displacing a more viscous fluid; in the inverse situation, with the more viscous displacing the other, the interface is stable and no instability is seen. Essentially the same effect occurs driven by gravity (without injection) if the interface is horizontal and separates two fluids of different densities, the heavier one being above the other: this is known as the Rayleigh-Taylor instability. In the rectangular configuration the system evolves until a single finger (the Saffman–Taylor finger) forms, whilst in the radial configuration the pattern grows forming fingers by successive tip-splitting.[4]

An example of viscous fingering in a Hele-Shaw cell.

Most experimental research on viscous fingering has been performed on Hele-Shaw cells, which consist of two closely spaced, parallel sheets of glass containing a viscous fluid. The two most common set-ups are the channel configuration, in which the less viscous fluid is injected at one end of the channel, and the radial configuration, in which the less viscous fluid is injected at the centre of the cell. Instabilities analogous to viscous fingering can also be self-generated in biological systems.[5]