Why the Falkner–Skan Equation is More Complex than Blasius

1 The Two Equations

The **Blasius equation** (flat plate, $\beta = 0$):

$$f''' + f\,f'' = 0,$$

and the **Falkner–Skan equation** ($\beta \neq 0$):

$$f''' + f\,f'' + \beta\big(1 - f'^{\,2}\big) = 0,$$

share the same boundary conditions $f'(0) = 0$, $f'(\infty) = 1$, $f(0) = f_w$. The only difference is the extra term $\beta(1 - f'^2)$, yet this addition creates substantial mathematical and numerical difficulties.

2 Mathematical Complexity

2.1 A Nonlinearity That Changes Character Across the Profile

In the Blasius equation the sole nonlinearity is ff'', which is smooth and well-behaved everywhere. The Falkner–Skan term $\beta(1 - f'^2)$ behaves differently at different locations in the profile:

- **Far field** ($f' \to 1$): the term vanishes, so Blasius and Falkner–Skan are asymptotically identical.
- **Near the wall** ($f' \approx 0$): the term contributes a constant β, acting as a source ($\beta > 0$) or a sink ($\beta < 0$) of streamwise momentum depending on the sign of the pressure gradient.

For a *favourable* gradient ($\beta > 0$) this source reinforces the profile, keeping it full and well-attached. For an *adverse* gradient ($\beta < 0$) the sink drains near-wall momentum, driving the profile toward an inflection point and eventually to separation ($f''(0) = 0$).

2.2 Multiple Solution Branches

For sufficiently negative β the Falkner–Skan equation admits *two distinct solution branches* for the same boundary conditions (first shown by Hartree, 1937). One branch has $f''(0) > 0$ (physically relevant, attached flow); the other has $f''(0) < 0$ (reversed near-wall flow). The Blasius equation has a unique solution for each f_w, so this ambiguity does not arise there.

2.3 Finite Existence Range

The Blasius equation has a valid solution for any finite suction level and fails only at a specific blowing threshold ($v_w^* = 0.619$). The Falkner–Skan equation couples the pressure gradient and transpiration: for a given $\beta < 0$ there is a *minimum suction* $f_w^{\min}(\beta) > 0$ required to maintain an attached solution, and a *maximum blowing* beyond which no steady solution exists. The solution domain is a two-dimensional region in the (β, f_w) plane rather than a simple threshold.

3 Numerical Complexity

3.1 Ill-Conditioned Shooting Problem

Both equations are solved by a shooting method: integrate from $\eta = 0$ with a guessed $f''(0)$ and adjust until $f'(\eta_{\max}) = 1$. For Blasius, the residual $f'(\eta_{\max}) - 1$ is a smooth, monotone function of $f''(0)$, making the bracket easy to find and ‘brentq‘ converges in a few iterations.

For adverse β near the separation limit the residual becomes nearly flat near the correct $f''(0)$:

- As $\beta \to -0.1988$ (for $f_w = 0$), $f''(0) \to 0$. Near this limit a small change in $f''(0)$ produces almost no change in $f'(\eta_{\max})$ over a short domain, so the solver must integrate to very large η before the profiles diverge enough to detect.
- The destabilising $\beta(1 - f'^2)$ term amplifies errors in $f''(0)$ *exponentially* as η grows, making the integration sensitive to step-size errors in the near-wall region.

3.2 Stiffness Near the Wall for Adverse Gradients

For $\beta < 0$ with mild suction, the near-wall layer is thin and the outer region evolves slowly. These two scales create a mild stiffness: the adaptive ODE solver must take small steps near $\eta = 0$ to resolve the sharp gradient in f'', while the outer region (where $f' \approx 1$) would allow large steps. Without a step-size cap the solver can overshoot in the near-wall region for adverse cases, landing on the wrong branch. This is the reason a fixed `max_step` was originally introduced in the code — it was compensating for this issue, at the cost of being unnecessarily slow in the favourable-β and Blasius cases where no such difficulty exists.

3.3 Extended Integration Domain Required

For the standard Blasius profile, $f' \approx 1$ to within 10^{-4} by $\eta \approx 6$. For adverse β near separation the profile is much thicker and f' does not approach 1 until $\eta \gtrsim 12$–14. A larger integration domain ($\eta_{\max}$) is therefore needed to correctly evaluate the boundary condition at infinity, increasing the cost of every function evaluation inside the shooting loop.

4 Summary

	Blasius ($\beta = 0$)	**Falkner–Skan** ($\beta \neq 0$)
Nonlinearity	ff'' only	$ff'' + \beta(1 - f'^2)$, changes sign
Solution uniqueness	Unique for each f_w	Two branches possible for $\beta < 0$
Existence condition	Single blow-off threshold	Region in (β, f_w) plane
Shooting residual	Smooth, monotone	Flat near separation; ill-conditioned
Near-wall stiffness	None	Present for adverse β
Domain size needed	$\eta_{\max} \approx 6$–8	Up to $\eta_{\max} \approx 12$–14