Porous Flat Plate: Exact NS Solution vs. Boundary Layer Theory

1 Problem Statement

Steady, 2D, incompressible, laminar flow of a Newtonian fluid (density ρ, viscosity μ, kinematic viscosity $\nu = \mu/\rho$) past an infinite porous flat plate at $y = 0$. The far-field velocity is U (at $y \to \infty$). The plate imposes a uniform wall-normal velocity $v = -v_0$ everywhere ($v_0 > 0$ for suction into the plate, $v_0 < 0$ for blowing out of the plate). Gravity and pressure gradients are neglected.

In the BL analysis of Section 3 we adopt the standard sign convention: $v_w > 0$ denotes blowing (fluid leaving the wall) and $v_w < 0$ denotes suction (fluid entering the wall), with the dimensionless suction–blowing parameter $v_w^* = (v_w/U)\sqrt{Re_x}$.

2 Exact Navier–Stokes Solution (Uniform v)

2.1 Simplification

Under the assumptions of 2D ($w = 0$, $\partial/\partial z = 0$), steady ($\partial/\partial t = 0$), and constant $v = -v_0$ (so $\partial v/\partial y = 0$), continuity gives

$$\frac{\partial u}{\partial x} = 0 \implies u = u(y) \text{ only.}$$

The x-momentum equation reduces to

$$\rho v \frac{du}{dy} = \mu \frac{d^2u}{dy^2} \quad \implies \quad \mu\, u'' + \rho v_0\, u' = 0.$$

2.2 General Solution

Assuming $u = e^{\lambda y}$ gives the characteristic equation

$$\mu\lambda^2 + \rho v_0\, \lambda = 0 \implies \lambda = 0 \quad \text{or} \quad \lambda = -\frac{\rho v_0}{\mu},$$

so the general solution is

$$u(y) = C_1 + C_2\, e^{-(\rho v_0/\mu)\, y}.$$

2.3 Boundary Conditions and Existence of Solution

The boundary conditions are $u(0) = 0$ (no-slip) and $u(\infty) = U$ (far-field).

Case 1: Suction ($v_0 > 0$, $v = -v_0 < 0$). The exponent $-(\rho v_0/\mu)\, y < 0$ for $y > 0$, so $e^{-(\rho v_0/\mu)\, y} \to 0$ as $y \to \infty$. Applying the BCs:

$$C_1 = U, \qquad C_2 = -U,$$

yielding the **exact closed-form solution**

$$\boxed{u(y) = U\big(1 - e^{-(\rho v_0/\mu)\, y}\big)\,.}$$

This is a valid, bounded velocity profile.

Case 2: Blowing ($v_0 < 0$, i.e. $v > 0$). Now the exponent $-(\rho v_0/\mu)\, y > 0$, so $e^{-(\rho v_0/\mu)\, y} \to \infty$ as $y \to \infty$. To keep u bounded we must set $C_2 = 0$, giving $u = C_1$ everywhere. Applying $u(0) = 0$ forces $C_1 = 0$, so $u \equiv 0$—contradicting $u(\infty) = U \neq 0$. **No physical solution exists.**

2.4 Physical Interpretation

With suction the wall draws fluid toward the plate, confining viscous effects near $y = 0$ and producing a thin, stable exponential profile. With blowing the wall pushes fluid away from the plate; viscous diffusion cannot establish a connection between the no-slip wall and the free stream, and the exponential grows without bound.

3 Boundary Layer Solution (Non-Uniform $v_w(x)$)

3.1 Similarity Transform and Governing Equation

For the BL analysis we introduce the Blasius similarity variables

$$\psi = \sqrt{2\nu U x}\, f(\eta), \qquad \eta = y\sqrt{\frac{U}{2\nu x}},$$

which automatically satisfies continuity, giving velocity components

$$u = U f'(\eta), \qquad v = \sqrt{\frac{\nu U}{2x}}(\eta f' - f)\,.$$

The wall-normal velocity at $y = 0$ ($\eta = 0$) is

$$v_w = -f(0)\sqrt{\frac{\nu U}{2x}} \propto x^{-1/2}.$$

This is the critical difference from the exact solution of Section 2: *the wall velocity is not constant but decays as $x^{-1/2}$*, allowing both u and v to vary with x and y through a single self-similar variable η.

Substituting into the BL-approximated x-momentum equation yields the **Blasius equation** (zero pressure gradient, $\beta = 0$):

$$f''' + f\,f'' = 0,$$

with boundary conditions

$$f'(0) = 0 \quad \text{(no-slip)}, \qquad f'(\infty) = 1 \quad \text{(far field)}, \qquad f(0) \neq 0 \quad \text{(suction or blowing)}.$$

Note that the equation itself is unchanged from the standard Blasius problem—only the value of $f(0)$ differs. The **suction–blowing parameter** encodes this:

$$v_w^* = \frac{v_w}{U}\sqrt{Re_x} = -\frac{f(0)}{\sqrt{2}}, \qquad Re_x = \frac{Ux}{\nu}.$$

The standard (impermeable) Blasius solution corresponds to $v_w^* = 0$, $f(0) = 0$.

3.2 Velocity Profiles: Effect of Suction and Blowing

Figure 1 shows numerically computed profiles $u/U = f'(\eta)$ for several values of v_w^* spanning strong suction to near blow-off. Each curve is a solution of the same Blasius ODE with a different value of $f(0) = -\sqrt{2}\,v_w^*$.

3.3 Suction ($v_w^* < 0$)

With suction the wall draws fluid *toward* the plate ($v_w < 0$), so $f(0) > 0$.

Effect on the profile. Stronger suction shifts the profile toward smaller η: the velocity rises from zero to U over a shorter wall-normal distance. This corresponds to a *thinner* BL and a *steeper* wall gradient $f''(0)$, hence larger wall shear stress

$$\tau_w = \mu\,\frac{\partial u}{\partial y}\bigg|_{y=0} = \mu U f''(0)\sqrt{\frac{U}{2\nu x}}.$$

Physical mechanism. Suction continuously removes low-momentum fluid from the BL and replaces it with higher-momentum fluid convected from the far field. This opposes the natural growth of the BL due to viscous diffusion and keeps the profile "attached" to the wall.

Asymptotic suction profile. In the limit of very strong, *uniform* suction ($v_w^* \to -\infty$ in BL terms, or equivalently the constant-v_0 exact solution of Section 2), the profile collapses to the simple exponential

$$\frac{u}{U} = 1 - e^{-v_0 y/\nu},$$

which has zero BL thickness growth ($d\delta/dx = 0$) and is linearly stable.

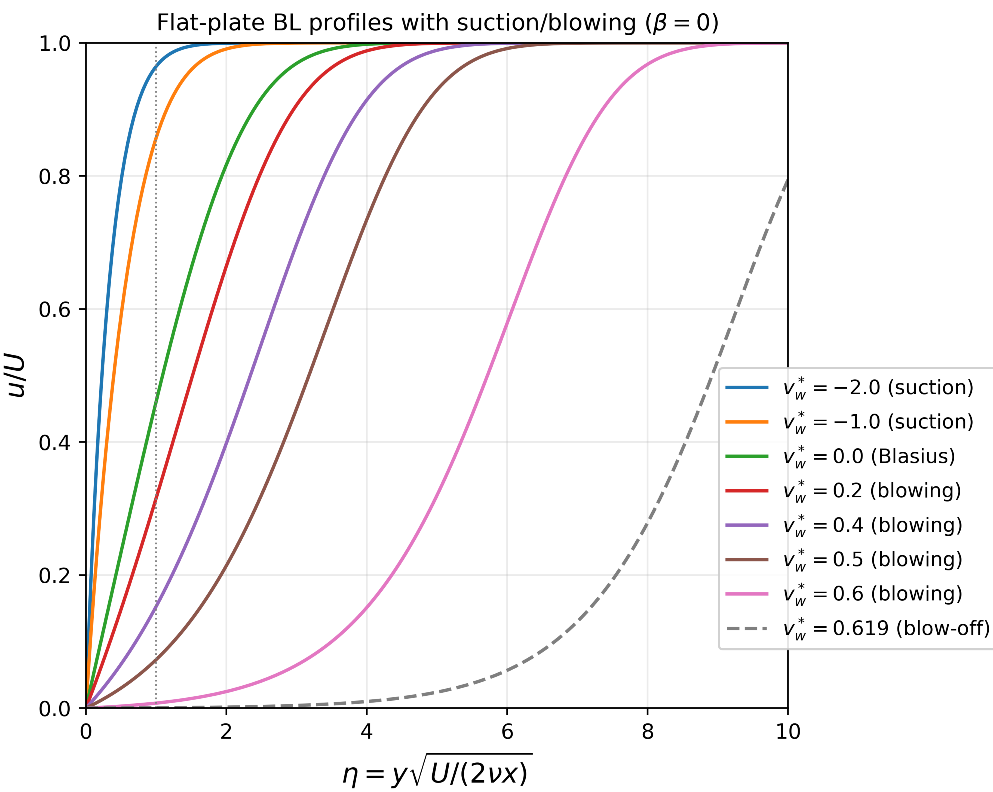

Figure 1: Flat-plate BL velocity profiles $u/U = f'(\eta)$ for suction ($v_w^* < 0$), impermeable Blasius ($v_w^* = 0$), and blowing ($v_w^* > 0$). Dashed curve: blow-off limit $v_w^* = 0.619$ at which $f''(0) = 0$.

Transition stability. Suction creates a *fuller* profile (no inflection point) analogous to a favorable pressure gradient. By Rayleigh's inflection-point criterion, such profiles are inviscidly stable and transition to turbulence is strongly delayed.

3.4 Blowing ($v_w^* > 0$)

With blowing the wall pushes fluid *away* from the plate ($v_w > 0$), so $f(0) < 0$.

Effect on the profile. Blowing shifts the profile to larger η: the BL thickens. The wall shear $f''(0)$ decreases, reducing τ_w. Crucially, for any $v_w^* > 0$ the profile develops an *inflection point* ($f'' = 0$ at some interior η), giving it an S-shape. This is visible in Figure 1 for the curves at $v_w^* = 0.4$ and 0.5.

Physical mechanism. Blowing injects low-momentum fluid into the BL from below. The BL must adjust to carry this additional mass flux, thickening and flattening the profile near the wall. The resulting inflection point is the signature of an adverse-pressure-gradient-like effect.

Transition stability. By Rayleigh's criterion, the inflection point in the blowing profiles makes them *inviscidly unstable.* They are susceptible to early transition to turbulence, similar to a flow with an adverse pressure gradient.

Blow-off limit. At $v_w^* = 0.619$, the shooting procedure yields $f''(0) = 0$:

$$v_w^* = 0.619 \implies \tau_w = 0.$$

The wall shear vanishes, the BL is said to be *blown off*, and the BL approximation breaks down entirely. For $v_w^* > 0.619$ no steady laminar similarity solution exists—consistent with the conclusion from the exact NS analysis that uniform blowing admits no physical solution.

Table 1 summarises the wall shear coefficient $f''(0)$ and the displacement-thickness integral for each case shown in Figure 1.

Table 1: Key quantities for the Blasius equation with suction/blowing. $\delta^*/\sqrt{2\nu x/U} = \int_0^\infty (1 - f')\, d\eta$.

v_w^*	$f(0)$	$f''(0)$	$\delta^*/\sqrt{2\nu x/U}$	Remark
-2.0	$+2.828$	1.882	0.398	Strong suction
-1.0	$+1.414$	1.030	0.648	Moderate suction
0.0	0.000	0.470	1.217	Standard Blasius
$+0.2$	-0.283	0.327	1.580	Mild blowing
$+0.4$	-0.566	0.191	2.160	Moderate blowing
$+0.5$	-0.707	0.117	2.620	Strong blowing
$+0.619$	-0.876	0.000	∞	Blow-off

3.5 Why the BL Solution Exists Where the Exact NS Solution Does Not

Unlike the exact (uniform-v) solution, the similarity solution is well-posed for blowing because:

- The wall-normal velocity is *non-uniform*: $v_w \propto x^{-1/2}$ decays downstream, so there is no global incompatibility between the no-slip condition and the far-field velocity.
- Both u and v vary with x *and* y through η, eliminating the constraint $u = u(y)$ that caused the breakdown in Section 2.
- The BL approximation requires $v_w \ll U$, i.e. blowing is treated as a perturbation, not as an $O(U)$ forcing.

The blow-off limit ($v_w^* = 0.619$) is nonetheless the BL's analogue of the exact solution's "no solution": beyond it, steady laminar flow cannot be maintained.

3.6 Comparison of Exact NS and BL Theory

	Exact NS (Sec. 2)	**BL Theory (Sec. 3)**
Wall-normal velocity	$v = \text{const}$	$v_w \propto x^{-1/2}$
Velocity field	$u = u(y)$ only	$u, v = f(\eta)$ via similarity
Suction solution	Exact: $U(1 - e^{-v_0 y/\nu})$	Thinner BL, larger $f''(0)$
Blowing solution	**None** (mathematically ill-posed)	Valid for $v_w^* < 0.619$
Breakdown mode	Any $v_0 < 0$	Physical blow-off at $v_w^* = 0.619$
Profile shape (blowing)	N/A	S-shaped with inflection point

4 Extension to $\beta \neq 0$: Falkner–Skan Equation

4.1 From Blasius to Falkner–Skan

The Blasius equation ($\beta = 0$) applies only to a flat plate with zero pressure gradient. For wedge flows or flows with an imposed streamwise pressure gradient the external velocity varies as $U(x) \propto x^m$, and the similarity analysis yields the **Falkner–Skan equation**:

$$f''' + f\,f'' + \beta\big(1 - f'^2\big) = 0,$$

where $\beta = 2m/(m+1)$ is the Hartree pressure-gradient parameter. The boundary conditions remain

$$f'(0) = 0, \quad f'(\infty) = 1, \quad f(0) = f_w \quad (f_w > 0 \text{ suction},\ f_w < 0 \text{ blowing}),$$

so the Blasius case ($\beta = 0$) is recovered exactly when $m = 0$. The sign and magnitude of β encode the local pressure gradient:

β	Flow type	Physical meaning
$\beta > 0$	Accelerating / wedge	Favourable pressure gradient, $dp/dx < 0$
$\beta = 0$	Flat plate	Zero pressure gradient (Blasius)
$\beta < 0$	Decelerating	Adverse pressure gradient, $dp/dx > 0$
$\beta = -0.1988$	Separation limit	$f''(0) = 0$ for impermeable wall ($f_w = 0$)

4.2 Effect of Pressure Gradient on Profile Shape

Favourable gradient ($\beta > 0$). The external flow accelerates, continuously supplying high-momentum fluid to the BL. Profiles are *fuller* (larger $f''(0)$, higher τ_w) with no inflection point. By Rayleigh's criterion such profiles are inviscidly stable, and transition is strongly delayed. The stagnation-point flow ($\beta = 1$, Hiemenz) represents the extreme case.

Zero gradient ($\beta = 0$). Standard Blasius profile with $f''(0) \approx 0.4696$.

Adverse gradient ($\beta < 0$). The external flow decelerates and the BL thickens rapidly. An inflection point appears in the profile, reducing $f''(0)$ and eventually driving it to zero at $\beta = -0.1988$ (Hartree separation criterion for $f_w = 0$). For $\beta < -0.1988$ no steady impermeable-wall solution exists—a result analogous to the blow-off condition in the $\beta = 0$ case.

5 Full NS Solution with the Similarity v-Profile

We now ask: if we impose the same v-profile as the BL case (i.e. use the stream-function), do the *full* Navier–Stokes equations reduce to something tractable?

5.1 Substitution into Full NS

With $u = Uf'(\eta)$ and $v = \sqrt{\nu U/(2x)}\,(\eta f' - f)$, we compute each term of the full (unapproximated) x-momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

Left-hand side.

$$u\frac{\partial u}{\partial x} = Uf' \cdot Uf''\left(-\frac{\eta}{2x}\right) = -\frac{U^2\eta f' f''}{2x},$$

$$v\frac{\partial u}{\partial y} = \sqrt{\frac{\nu U}{2x}}(\eta f' - f) \cdot Uf''\sqrt{\frac{U}{2\nu x}} = \frac{U^2 f''(\eta f' - f)}{2x}.$$

Adding:

$$\text{LHS} = \frac{U^2 f''}{2x}[-\eta f' + \eta f' - f] = -\frac{U^2 f f''}{2x}.$$

Right-hand side. The y-derivative term:

$$\nu \frac{\partial^2 u}{\partial y^2} = \nu \cdot U f''' \cdot \frac{U}{2\nu x} = \frac{U^2 f'''}{2x}.$$

The x-derivative term $\nu\,\partial^2 u/\partial x^2$ involves derivatives of $\eta f''/x$ with respect to x. A direct calculation shows

$$\nu \frac{\partial^2 u}{\partial x^2} = \frac{\nu U(3\eta f'' + \eta^2 f''')}{4x^2} = \mathcal{O}\left(\frac{1}{Re_x}\right) \frac{U^2 f'''}{2x},$$

where $Re_x = Ux/\nu \gg 1$ in the boundary layer regime.

5.2 Result: Blasius Equation + Small Corrections

Collecting terms and multiplying through by $2x/U^2$:

$$\boxed{f''' + f\,f'' = \mathcal{O}\left(\frac{1}{Re_x}\right).}$$

For $Re_x \gg 1$ the right-hand side is negligible and we recover the **Blasius equation** exactly. Therefore:

1. The similarity makes the problem *well-posed* for the full NS equations—unlike the uniform-blowing case, which has no solution.
2. The BL (Blasius) solution is an asymptotic solution of the full NS equations correct to $\mathcal{O}(Re_x^{-1})$.
3. Blow-off at $v_w^* = 0.619$ remains the physical limit; beyond that point even the full NS problem with the similarity v-profile lacks a steady laminar solution.

5.3 Key Insight

Assuming $v \propto x^{-1/2}$ (rather than $v = \text{const}$) removes the root cause of the breakdown identified in Section 2. The x-dependence allows u to vary with x as well, breaking the constraint $u = u(y)$ and permitting a self-consistent, bounded solution for moderate blowing—one that the full NS equations confirm, up to corrections of order $1/Re_x$.