Appendix 9.3 — A

A.3 Spherical Coordinates

In a three-dimensional spherical coordinate system r, θ, ϕ with corresponding unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$, the gradient of a scalar Φ is

$$\nabla \Phi = \frac{\partial \Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}\hat{\boldsymbol{\phi}}, \tag{A.14}$$

and its Laplacian is

$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2}. \tag{A.15}$$

The divergence of a vector $\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_\phi\hat{\boldsymbol{\phi}}$ is

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_\theta \sin\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}, \tag{A.16}$$

and the curl of $\mathbf{v}$ is

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left(\frac{\partial v_\phi \sin\theta}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial r v_\phi}{\partial r}\right)\hat{\boldsymbol{\theta}}$$
$$+\frac{1}{r}\left(\frac{\partial r v_\theta}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)\hat{\boldsymbol{\phi}}. \tag{A.17}$$

The incompressible Navier-Stokes equation for velocity components u_r, u_θ, u_ϕ, apart from a gravitational force, is

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2}{r} - \frac{u_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$+ \nu\left(\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial u_\theta \sin\theta}{\partial \theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right), \tag{A.18}$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta}{r} - \frac{u_\phi^2 \cot\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$
$$+ \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial \phi}\right), \tag{A.19}$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r u_\phi}{r} + \frac{u_\theta u_\phi \cot\theta}{r} = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi}$$
$$+ \nu\left(\nabla^2 u_\phi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2\sin^2\theta} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial \phi}\right). \tag{A.20}$$