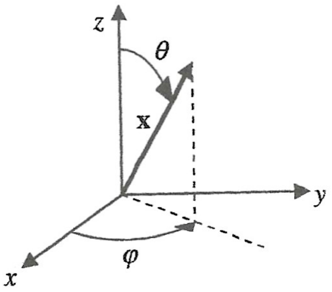
Appendix 9.2-A

Spherical Coordinates (Figure B.3)



Position: $\mathbf{x} = (r, \theta, \varphi) = r\mathbf{e}_r$; $x = r\cos\varphi\sin\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\theta$; or

$r = \sqrt{x^2+y^2+z^2}$, $\theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right)$, and $\varphi = \tan^{-1}(y/x)$

Unit vectors: $\mathbf{e}_r = \mathbf{e}_x\sin\theta\cos\varphi + \mathbf{e}_y\sin\theta\sin\varphi + \mathbf{e}_z\cos\theta$,

$\mathbf{e}_\theta = \mathbf{e}_x\cos\theta\cos\varphi + \mathbf{e}_y\cos\theta\sin\varphi - \mathbf{e}_z\sin\theta$, $\mathbf{e}_\varphi = -\mathbf{e}_x\sin\varphi + \mathbf{e}_y\cos\varphi$

Unit vector dependencies: $\partial\mathbf{e}_r/\partial r = 0$, $\partial\mathbf{e}_r/\partial\theta = \mathbf{e}_\theta$, $\partial\mathbf{e}_r/\partial\varphi = \mathbf{e}_\varphi\sin\theta$

$\partial\mathbf{e}_\theta/\partial r = 0$, $\partial\mathbf{e}_\theta/\partial\theta = -\mathbf{e}_r$, $\partial\mathbf{e}_\theta/\partial\varphi = \mathbf{e}_\varphi\cos\theta$

$\partial\mathbf{e}_\varphi/\partial r = 0$, $\partial\mathbf{e}_\varphi/\partial\theta = 0$, $\partial\mathbf{e}_\varphi/\partial\varphi = -\mathbf{e}_r\sin\theta - \mathbf{e}_\theta\cos\theta$

Gradient Operator: $\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$

Surface integral, S, of $f(r, \theta, \varphi)$ over the sphere defined by $r = \xi$: $S = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} f(\xi, \theta, \varphi)\xi^2\sin\theta\, d\varphi\, d\theta$

Surface integral, S, of $f(r, \theta, \varphi)$ over the cone defined by $\theta = \psi$: $S = \int_{r=0}^{+\infty}\int_{\varphi=0}^{2\pi} f(r, \psi, \varphi) r\sin\psi\, d\varphi\, dr$

Surface integral, S, of $f(r, \theta, \varphi)$ over the half plane defined by $\varphi = \zeta$: $S = \int_{r=0}^{+\infty}\int_{\theta=0}^{\pi} f(r, \theta, \zeta) r\, d\theta\, dr$

Volume integral, V, of $f(r, \theta, \varphi)$ over all space: $V = \int_{r=0}^{+\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} f(r, \theta, \varphi) r^2\sin\theta\, d\varphi\, d\theta\, dr$

Spherical Coordinates (Figure B.3)

Position and velocity vectors: $\mathbf{x} = (r, \theta, \varphi) = r\mathbf{e}_r; \mathbf{u} = (u_r, u_\theta, u_\varphi) = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_\varphi\mathbf{e}_\varphi$

Gradient of a scalar ψ: $\nabla\psi = \mathbf{e}_r\dfrac{\partial\psi}{\partial r} + \mathbf{e}_\varphi\dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} + \mathbf{e}_\varphi\dfrac{1}{r\sin\theta}\dfrac{\partial\psi}{\partial\varphi}$

Laplacian of a scalar ψ: $\nabla^2\psi = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\psi}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial\psi}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2\psi}{\partial\varphi^2}$

Divergence of a vector: $\nabla\cdot\mathbf{u} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2u_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial(u_\theta\sin\theta)}{\partial\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial u_\varphi}{\partial\varphi}$

Curl of a vector, vorticity: $\boldsymbol{\omega} = \nabla\times\mathbf{u} = \dfrac{\mathbf{e}_r}{r\sin\theta}\left(\dfrac{\partial(u_\varphi\sin\theta)}{\partial\theta} - \dfrac{\partial u_\theta}{\partial\varphi}\right) + \dfrac{\mathbf{e}_\theta}{r}\left(\dfrac{1}{\sin\theta}\dfrac{\partial u_r}{\partial\varphi} - \dfrac{\partial(ru_\varphi)}{\partial r}\right)$
$+\dfrac{\mathbf{e}_\varphi}{r}\left(\dfrac{\partial(ru_\theta)}{\partial r} - \dfrac{\partial u_r}{\partial\theta}\right)$

Laplacian of a vector:

$$\nabla^2\mathbf{u} = \mathbf{e}_r\left(\nabla^2u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right)$$
$$+\,\mathbf{e}_\theta\left(\nabla^2u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right)$$
$$+\,\mathbf{e}_\varphi\left(\nabla^2u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right)$$

Strain rate S_{ij} and viscous stress τ_{ij} for an incompressible fluid where $\tau_{ij} = 2\mu S_{ij}$:

$$S_{rr} = \frac{\partial u_r}{\partial r} = \frac{1}{2\mu}\tau_{rr},\quad S_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} = \frac{1}{2\mu}\tau_{\theta\theta},\quad S_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r} = \frac{1}{2\mu}\tau_{\varphi\varphi},$$
$$S_{\theta\varphi} = \frac{\sin\theta}{2r}\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right) + \frac{1}{2r\sin\theta}\frac{\partial u_\theta}{\partial\varphi} = \frac{1}{2\mu}\tau_{\theta\varphi},\quad S_{\varphi r} = \frac{1}{2r\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right) = \frac{1}{2\mu}\tau_{\varphi r},$$
$$S_{r\theta} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial\theta} = \frac{1}{2\mu}\tau_{r\theta}$$

Equation of continuity:

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\rho u_\varphi) = 0$$

Navier-Stokes equations with constant ρ, constant ν, and no body force:

$$\frac{\partial u_r}{\partial t} + (\mathbf{u}\cdot\nabla)u_r - \frac{u_\theta^2 + u_\varphi^2}{r}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2 \sin\theta}\frac{\partial(u_\theta \sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right],$$

$$\frac{\partial u_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2 \cot\theta}{r}$$
$$= -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left[\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right],$$

$$\frac{\partial u_\varphi}{\partial t} + (\mathbf{u}\cdot\nabla)u_\varphi + \frac{u_\varphi u_r}{r} + \frac{u_\theta u_\varphi \cot\theta}{r}$$
$$= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} + \nu\left[\nabla^2 u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right]$$

where

$$\mathbf{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi},$$
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$