

Appendix D – 3D Potential Flows

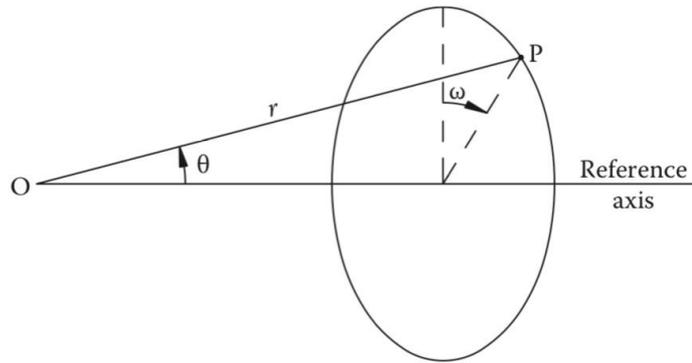


FIGURE 5.1
Definition sketch of spherical coordinates.

3D axisymmetric bodies of interest

Spherical coordinates: $P = P(r, \theta, \omega)$ with $u_\omega = 0$ and $\partial/\partial\omega = 0$

$$\underline{u} = \nabla\varphi \quad \nabla^2\varphi = 0 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \varphi_r) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \varphi_\theta)$$

$$u_r = \varphi_r$$

$$u_\theta = \frac{1}{r} \varphi_\theta$$

$$\nabla \cdot \underline{u} = 0 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (u_\theta \sin\theta)$$

Stokes stream function

$$u_r = \frac{1}{r^2 \sin\theta} \psi_\theta$$

$$u_\theta = -\frac{1}{r \sin\theta} \psi_r$$

Continuity identically satisfied.

General solution of $\nabla^2\varphi = 0$ by separation of variables provides fundamental solutions that can be combined to obtain solutions about simple geometries: uniform stream, source/sink, doublet, blunt nose, sphere, etc. using spherical harmonics: https://en.wikipedia.org/wiki/Spherical_harmonics.

$$\varphi(r, \theta) = R(r)T(\theta) \quad \text{since } \neq f(\omega)$$

Substitution $\nabla^2\varphi = 0$:

$$\frac{T}{r^2} \frac{d}{dr} (r^2 R_r) + \frac{R}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta T_\theta) = 0$$

$$\times \frac{r^2}{RT} \Rightarrow \underbrace{\frac{1}{R} \frac{d}{dr} (r^2 R_r)}_{f(r)} = - \underbrace{\frac{1}{T \sin \theta} \frac{d}{d\theta} (\sin \theta T_\theta)}_{f(\theta)}$$

$$\therefore \text{LHS} = \text{RHS} = \text{constant} = \underbrace{l(l+1)}_{\text{for ease } T(\theta) \text{ solution}}$$

$$\frac{1}{R} \frac{d}{dr} (r^2 R_r) = l(l+1) \quad \text{equidimensional equation:}$$

$$R(r) = kr^\alpha$$

$$\alpha(\alpha+1)kr^\alpha - l(l+1)kr^\alpha = 0$$

$$\text{i.e., } \alpha = l \text{ and } \alpha = -(l+1)$$

$$\therefore R_l(r) = A_l r^l + \frac{B_l}{r^{l+1}}$$

R_l valid any l for arbitrary constants A_l and B_l .

$$\frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta T_\theta) + l(l+1)T = 0$$

Legendre equation can be reduced to standard form with $x = \cos \theta$

$$\frac{d}{dx} [(1-x^2) T_x] + l(l+1)T = 0$$

Solution: Legendre functions 1st kind $P_l(x)$ and 2nd kind $Q_l(x)$ such that

$$T_l(\theta) = C_l P_l(\cos \theta) + D_l Q_l(\cos \theta)$$

$Q_l(\cos \theta)$ diverges $\cos \theta = \pm 1 \therefore D_l = 0$

$P_l(\cos \theta)$ diverges $\cos \theta = \pm 1$ for no singularities in flow field

unless $l = \text{integer} \therefore l = \text{integer}$

$$\varphi_l(r, \theta) = \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta) \quad C_l \text{ absorb into } A_l \text{ and } B_l$$

Fundamental
solutions,
superimposed for
additional solutions

Since $\nabla^2 \varphi$ is linear: $\phi(r, \theta) = \sum_{l=0}^{\infty} \phi_l(r, \theta)$

where $P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$ Legendre polynomial

$P_0(x) = 1$ order I

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2} (3x^2 - 1)$$

etc.

$$\text{uniform flow: } B_l = 0, A_l = 0 \quad l \neq 1, \\ = U \quad l = 1$$

$$P_l(\cos \theta) = \cos \theta$$

$$\varphi(r, \theta) = Ur \cos \theta = Ux \quad x = r \cos \theta$$

$$u_r = \varphi_r = U \cos \theta = \frac{\psi_\theta}{r^2 \sin \theta}$$

$$\psi = \frac{1}{2} Ur^2 \sin^2 \theta \pm f(r)$$

$$u_\theta = \frac{1}{r} \varphi_\theta = -U \sin \theta = -\frac{1}{r \sin \theta} \psi_r$$

$$\psi = \frac{1}{2} Ur^2 \sin^2 \theta \pm g(\theta) \quad f(r) = g(\theta) = \text{constant} = 0$$

An alternative way of evaluating $\psi(r, \theta)$ is simply to invoke its definition. Then, considering an arbitrary point P in the fluid as shown in Figure 5.3, the amount of fluid crossing the surface generated by OP due to the uniform flow will be $2\pi\psi$. However, the flow area perpendicular to the velocity vector is $\pi(r \sin \theta)$. Hence, it follows from the definition of ψ that

$$2\pi\psi = U\pi (r \sin \theta)^2$$

or

$$\psi(r, \theta) = \frac{1}{2} Ur^2 \sin^2 \theta.$$

$$2\pi\psi = dQ = U dA \\ = U\pi(r \sin \theta)^2$$

This agrees with the result obtained by the other method.

Both the methods outlined above for evaluating the stream function are useful, and each will be used in the following sections. The particular method employed will depend upon the complexity of the problem, and it is evident that the second method can be conveniently employed only for very simple flow fields.

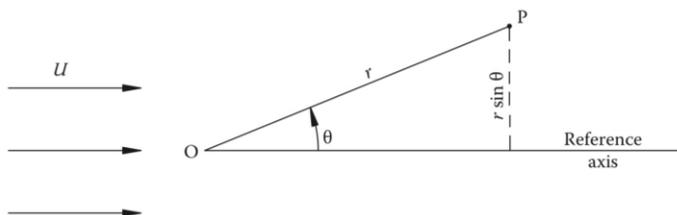


FIGURE 5.3
Geometry for evaluating the stream function for a uniform flow.

Source/sink: $A_l = 0$ for all l

$B_l = 0$ for $l \neq 0$

$B_l = B_0 \neq 0$ for $l = 0$

$P_0(\cos \theta) = 1$

$$\varphi(r, \theta) = B_0/r \quad Q = \int_S \underline{u} \cdot \underline{n} dS = -4\pi B_0 \quad \varphi = -Q/(4\pi r)$$

$$u_r = -B_0/r^2$$

$$u_\theta = 0$$

$$\psi(r, \theta) = -\frac{Q}{4\pi} (1 + \cos \theta)$$

Doublet: $\varphi(r, \theta) = \frac{Q}{4\pi r} \left[\frac{\delta x}{r} \cos \theta (1 + O\left(\frac{\delta r}{r}\right)) \right]$ $\delta x \rightarrow 0$ and $Q \rightarrow \infty$ such that $Q \delta x \rightarrow \mu$

$$\varphi(r, \theta) = \frac{\mu}{4\pi r^2} \cos \theta$$

$$u_r = \varphi_r = -\frac{\mu}{2\pi r^3} \cos \theta = \frac{1}{r^2 \sin \theta} \psi_\theta$$

$$u_\theta = \frac{1}{r} \varphi_\theta = -\frac{\mu}{4\pi r^3} \sin \theta = -\frac{1}{r \sin \theta} \psi_r$$

$$\psi = -\frac{\mu}{4\pi r} \sin^2 \theta$$

Sphere: $\psi(r, \theta) = \frac{1}{2} U r^2 \sin^2 \theta - \frac{\mu}{4\pi r} \sin^2 \theta = \text{uniform stream} + \text{doublet}$

$$\psi = 0 \quad r = r_0 \Rightarrow r_0 = \left(\frac{\mu}{2\pi U} \right)^{1/3} \quad \text{i.e. } \mu = 2\pi U a^3$$

$$\psi(r, \theta) = \frac{1}{2} U \left(r^2 - \frac{a^3}{r} \right) \sin^2 \theta$$

$$\varphi(r, \theta) = U \left(r + \frac{1}{2} \frac{a^3}{r^2} \right) \cos \theta$$

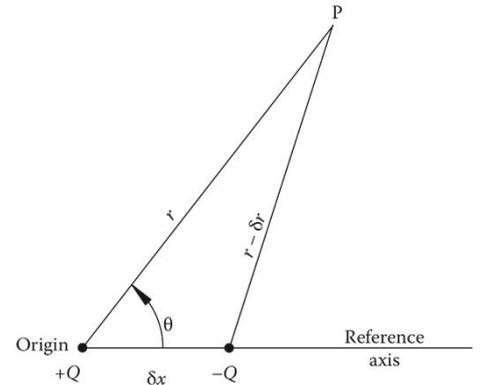


FIGURE 5.5 Superposition of a source and a sink that become a doublet as $\delta x \rightarrow 0$.

$$u_r = U \cos \theta \left(1 - \left(\frac{a}{r} \right)^3 \right)$$

$$u_\theta = -U \sin \theta \left(1 + \frac{1}{2} \left(\frac{a}{r} \right)^3 \right)$$

$$u_r(r = a) = 0$$

$$u_\theta(r = a) = -U \sin \theta \left(\frac{3}{2} \right)$$

stagnation points: $\theta = 0, \pi$

$$\max u_\theta = \pm \pi/2$$

$$u_{\theta_{max}} = \pm 1.5 U$$

$$C_p = 1 - \frac{9}{4} \sin^2 \theta$$

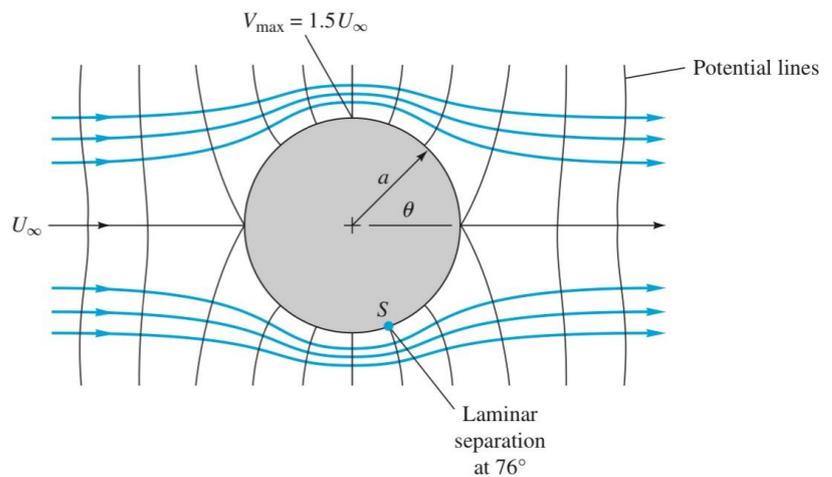


Fig. 8.31 Streamlines and potential lines for inviscid flow past a sphere.

Recall cylinder: $u_\theta = 2U \sin \theta$ $C_p = 1 - 4 \sin^2 \theta$