Stokes stream function for axisymmetric flow

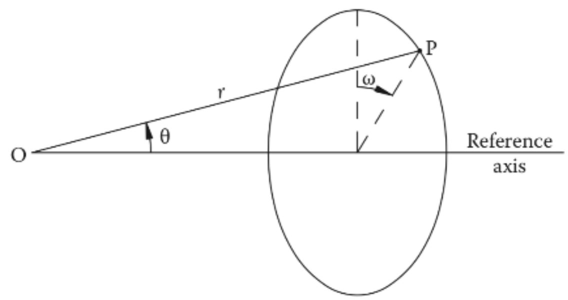

FIGURE 5.1
Definition sketch of spherical coordinates.

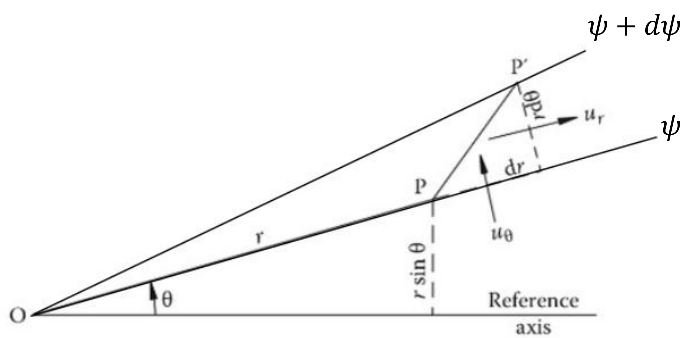

FIGURE 5.2
Velocity components and flow areas defined by a reference point P and neighboring point P′.

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta \sin\theta) = 0$$ Continuity equation

$\nabla \cdot \underline{u} = 0$ for $\frac{\partial}{\partial\phi} = 0$

Identically satisfied for $u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$
where $\psi(r,\theta)$ is Stokes stream function.

Let OP rotate around the reference axis such that ω varies by 2π while r, θ are fixed. The amount of fluid that crosses the surface of revolution formed by OP is

πD = circumference circle generated OP rotated

$$2\pi\, d\psi = \overbrace{2\pi r \sin\theta}^{\pi D} \underbrace{(u_r\, r\, d\theta - u_\theta\, dr)} = dQ$$

Quantity fluid per unit area, i.e., outflow − inflow

$d\psi = u_r r^2 \sin\theta \; d\theta - u_\theta r \sin\theta \; dr \; = \; dQ/2\pi$ = equals difference in flow rate per unit radian with units m^3/s.

Note in 2D plane flow ψ has units m^2/s.

$$d\psi = \frac{\partial\psi}{\partial\theta} d\theta + \frac{\partial\psi}{\partial r} dr$$

i.e., $\frac{\partial\psi}{\partial\theta} = u_r r^2 \sin\theta$, $\qquad \frac{\partial\psi}{\partial r} = -u_\theta r \sin\theta$

such that:

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta} \quad \text{and} \quad u_\theta = -\frac{1}{r \sin\theta} \frac{\partial\psi}{\partial r}$$