Bernard: Fluid Dynamics

Vorticity and Vector Potential:

$$\nabla \times \underline{u} = \underline{\omega} \quad (1) \qquad \underline{u} = u_r \hat{e}_r + u_\theta \hat{e}_\theta = \nabla \times \underline{A} \quad (2)$$

where $\underline{A} = \left(0, 0, \frac{\psi}{r \sin\theta}\right)$ is the vector potential.
For axisymmetric flow in spherical coordinates, note $\nabla \cdot \underline{A} = 0$.

$$\nabla \times (\nabla \times \underline{A}) = \underline{\omega} \quad (3)$$

In Helmholtz decomposition, i.e., $\nabla \underline{w} = \nabla \times \underline{A}$.

$$\nabla \times \left(\nabla \times \frac{\psi}{r \sin\theta} \hat{e}_\phi\right) = \underline{\omega} \quad (4)$$

Applying the curl operator in spherical coordinates:

$$-\frac{1}{r \sin\theta} L\psi \hat{e}_\phi = \omega_\phi \hat{e}_\phi$$

where the operator L is defined as:

$$L\psi = \psi_{rr} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \psi_\theta\right) \quad (5)$$

using definition curl in spherical coordinates or $\nabla^2 \left(\frac{\psi}{r \sin\theta}\right) \hat{e}_\phi$.

Proof

In axisymmetric spherical coordinates (r, θ, ϕ), the vector potential $\underline{A}$ is defined as:

$$\underline{A} = A_\phi \hat{e}_\phi = \frac{\psi}{r \sin\theta} \hat{e}_\phi$$

The vorticity $\underline{\omega}$ is given by the relation $\underline{\omega} = \nabla \times (\nabla \times \underline{A})$. We aim to show that:

$$\omega_\phi = -\frac{1}{r \sin\theta} L\psi, \quad \text{where } L\psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial\psi}{\partial\theta}\right)$$

1 Approach 1: Direct Curl Calculation

The velocity is defined as $\underline{u} = \nabla \times \underline{A}$. In spherical coordinates, for $\underline{A} = (0, 0, A_\phi)$:

$$\underline{u} = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) \hat{e}_r - \frac{\partial}{\partial r}(r A_\phi) \hat{e}_\theta\right]$$

Substituting $A_\phi = \frac{\psi}{r \sin\theta}$:

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{1}{r \sin\theta} \frac{\partial\psi}{\partial r}$$

Now, calculate the vorticity $\underline{\omega} = \nabla \times \underline{u}$. For axisymmetric flow, only the ϕ-component survives:

$$\omega_\phi = \frac{1}{r} \left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right]$$

Substituting the velocity components:

$$\begin{aligned}\omega_\phi &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial r}\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right] \\ &= -\frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^3}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\end{aligned}$$

Factoring out $-\frac{1}{r\sin\theta}$ yields the desired operator L:

$$\omega_\phi = -\frac{1}{r\sin\theta}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right] = -\frac{1}{r\sin\theta}L\psi$$

2 Approach 2: Vector Laplacian Identity

For a solenoidal vector field ($\nabla\cdot\underline{A}=0$), we use the identity:

$$\nabla\times(\nabla\times\underline{A}) = -\nabla^2\underline{A}$$

In curvilinear coordinates, the ϕ-component of the vector Laplacian is:

$$(\nabla^2\underline{A})_\phi = \nabla^2 A_\phi - \frac{A_\phi}{r^2\sin^2\theta}$$

Expanding the scalar Laplacian $\nabla^2 A_\phi$ and substituting $A_\phi = \frac{\psi}{r\sin\theta}$:

$$\begin{aligned}\nabla^2\left(\frac{\psi}{r\sin\theta}\right) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{\psi}{r\sin\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\frac{\psi}{r\sin\theta}\right) \\ &= \frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial\psi}{\partial\theta} - \psi\cot\theta\right)\end{aligned}$$

After collecting terms and subtracting $\frac{\psi}{r^3\sin^3\theta}$, the expression simplifies to:

$$-\nabla^2\underline{A} = -\frac{1}{r\sin\theta}\left[\psi_{rr} + \frac{1}{r^2}\psi_{\theta\theta} - \frac{\cot\theta}{r^2}\psi_\theta\right]\hat{e}_\phi = -\frac{1}{r\sin\theta}L\psi\hat{e}_\phi$$

This confirms that $\omega_\phi\hat{e}_\phi = \nabla^2\left(\frac{\psi}{r\sin\theta}\right)\hat{e}_\phi$ as noted in the derivation.

Stokes Flow: $\nabla^2\underline{\omega} = 0$

$$\nabla\times\nabla\times\underline{\omega} = \nabla(\nabla\cdot\underline{\omega}) - \nabla^2\underline{\omega}$$

Since $\nabla\cdot\underline{\omega} = 0$:

$$\begin{aligned}\nabla\times(\nabla\times\underline{\omega}) &= 0 = -\nabla^2\underline{\omega} \\ \nabla\times(\nabla\times\omega_\phi\hat{e}_\phi) &= 0 = -\nabla^2\omega_\phi\hat{e}_\phi \\ \nabla\times\left(\nabla\times\frac{-L\psi}{r\sin\theta}\hat{e}_\phi\right) &= 0 = -\nabla^2\omega_\phi\hat{e}_\phi\end{aligned}$$

We previously established the identity (4), which states that for any scalar function f:

$$\nabla\times\left(\nabla\times\frac{f}{r\sin\theta}\hat{e}_\phi\right) = -\frac{1}{r\sin\theta}L(f)\hat{e}_\phi$$

By choosing $f = L\psi$ (noting that $\omega_\phi \propto L\psi$), we substitute this into the Stokes flow condition:

$$\nabla \times \left(\nabla \times \left[\frac{-L\psi}{r \sin\theta} \hat{e}_\phi \right] \right) = -\frac{1}{r \sin\theta} L(-L\psi) \hat{e}_\phi = 0$$

This leads directly to the fourth-order governing equation for the stream function:

$$L(L\psi) = 0 \quad \text{or} \quad L^2\psi = 0$$

Expanding the operator:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \right) \right]^2 \psi = 0$$

Boundary Conditions (BC):

- **No-slip at surface:** $(u_r, u_\theta) = (0, 0)$ at $r = a$
- i.e., $\psi_\theta(a, \theta) = 0$ and $\psi_r(a, \theta) = 0$
- **Uniform flow at infinity:** As $r \to \infty$, $\psi = \frac{1}{2} U r^2 \sin^2\theta$

Velocity Relations:

$$u_r = \frac{\psi_\theta}{r^2 \sin\theta}, \quad u_\theta = \frac{-\psi_r}{r \sin\theta}$$