Mathematical Proof of the Velocity Identity via Euler's Theorem

Derivation of $-u_i = x_j \frac{\partial u_i}{\partial x_j}$

1 Introduction

Given the velocity component defined as:

$$u_i = c\left[\frac{x}{r^3}x_i + \frac{\delta_{i1}}{r}\right] \tag{1}$$

where $r = \sqrt{x_j x_j}$, we aim to prove that $-u_i = x_j \frac{\partial u_i}{\partial x_j}$ using two distinct methods.

2 Approach 1: Euler's Theorem for Homogeneous Functions

Euler's Theorem states that if a function $f(\mathbf{x})$ is homogeneous of degree n, such that $f(\lambda\mathbf{x}) = \lambda^n f(\mathbf{x})$, then:

$$x_j \frac{\partial f}{\partial x_j} = nf \tag{2}$$

Step 1: Determine the Degree of Homogeneity

Let us scale the coordinates by a factor λ, noting that $r(\lambda\mathbf{x}) = \lambda r(\mathbf{x})$ and $x(\lambda\mathbf{x}) = \lambda x$. Substituting into u_i:

$$\begin{aligned} u_i(\lambda\mathbf{x}) &= c\left[\frac{(\lambda x)}{(\lambda r)^3}(\lambda x_i) + \frac{\delta_{i1}}{(\lambda r)}\right] \\ &= c\left[\frac{\lambda^2 x x_i}{\lambda^3 r^3} + \frac{\delta_{i1}}{\lambda r}\right] \\ &= \lambda^{-1}\left(c\left[\frac{x x_i}{r^3} + \frac{\delta_{i1}}{r}\right]\right) \\ &= \lambda^{-1} u_i(\mathbf{x}) \end{aligned}$$

The function u_i is homogeneous of degree $n = -1$.

Step 2: Apply the Theorem

Substituting $n = -1$ into the Euler identity:

$$x_j \frac{\partial u_i}{\partial x_j} = (-1)u_i = -u_i \tag{3}$$

This completes the proof via the homogeneity property.

3 Approach 2: Direct Differentiation

We define $u_i = A_i + B_i$, where $A_i = c\frac{x x_i}{r^3}$ and $B_i = c\frac{\delta_{i1}}{r}$. Recall that $\frac{\partial r}{\partial x_j} = \frac{x_j}{r}$.

Step 1: Differentiate the first term A_i

Using the product rule on $A_i = c(x)(x_i)(r^{-3})$:

$$\begin{aligned}\frac{\partial A_i}{\partial x_j} &= c\left[\frac{\partial x}{\partial x_j}\frac{x_i}{r^3} + x\frac{\partial x_i}{\partial x_j}\frac{1}{r^3} + xx_i\frac{\partial (r^{-3})}{\partial x_j}\right] \\ &= c\left[\frac{\delta_{1j}x_i}{r^3} + \frac{x\delta_{ij}}{r^3} - \frac{3xx_ix_j}{r^5}\right]\end{aligned}$$

Multiplying by x_j:

$$\begin{aligned}x_j\frac{\partial A_i}{\partial x_j} &= c\left[\frac{xx_i}{r^3} + \frac{xx_i}{r^3} - \frac{3xx_i(x_jx_j)}{r^5}\right] \\ &= c\left[\frac{2xx_i}{r^3} - \frac{3xx_ir^2}{r^5}\right] = -\frac{cxx_i}{r^3} = -A_i\end{aligned}$$

Step 2: Differentiate the second term B_i

$$\frac{\partial B_i}{\partial x_j} = c\delta_{i1}\left(-\frac{1}{r^2}\frac{x_j}{r}\right) = -\frac{c\delta_{i1}x_j}{r^3}$$

Multiplying by x_j:

$$x_j\frac{\partial B_i}{\partial x_j} = -\frac{c\delta_{i1}(x_jx_j)}{r^3} = -\frac{c\delta_{i1}r^2}{r^3} = -\frac{c\delta_{i1}}{r} = -B_i$$

Conclusion

Summing the results:

$$x_j\frac{\partial u_i}{\partial x_j} = -A_i - B_i = -u_i \tag{4}$$