Non-inertial Frame of Reference

NS equations: derived for inertial reference frame i.e. stationary or moving at constant speed relative stationary reference frame; for the present purposes stationary reference frame = stationary with respect to distant stars, which is appropriate for geophysical flows. However, often for engineering applications such as ship motions the earth is taken as the inertial frame and vehicle fixed as the non-inertial frame.

$\nabla \cdot \underline{u} = 0 \quad \rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u}$ incompressible flow

note: $\mu \nabla^2 u_i = 2\mu \frac{\partial S_{ij}}{\partial x_i} = \mu \frac{\partial}{\partial x_i}\left(u_{i,j} + u_{j,i}\right) = -\mu\, \varepsilon_{jik} \frac{\partial \omega_k}{\partial x_i}$

or $\mu \nabla^2 \underline{u} = -\mu \nabla \times \underline{\omega}$

(1) Seeming paradox that net viscous force $f(\underline{\omega})$, even though postulate #2 for $\tau_{ij} = f(\varepsilon_{ij})$ states that rigid body rotation causes no shear stress, is resolved since involves derivatives of ω_k.

(2) $\mu \nabla^2 \underline{u} = 0$ when $\underline{\omega} =$ constant (solid body rotation), which also requires $\frac{\partial S_{ij}}{\partial x_j} = 0$.

Many applications require non-inertial reference frames: rotating machinery, maneuvering vehicles, geophysical flows (atmospheric, oceanic), etc.

The continuity equation is not altered, but the NS is.

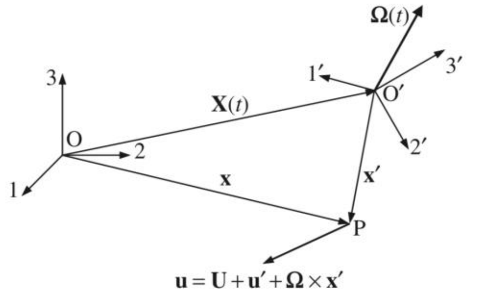


FIGURE 4.6 Geometry showing the relationship between a stationary coordinate system O123 and a noninertial coordinate system O′1′2′3′ that is moving, accelerating, and rotating with respect to O123. In particular, the vector connecting O and O′ is **X**(t) and the rotational velocity of O′1′2′3′ is **Ω**(*t*). The vector velocity **u** at point P in O123 is shown. The vector velocity **u**′ at point P in O′1′2′3′ differs from u because of the motion of O′1′2′3′.

Non-inertial reference frame $O'1'2'3'$: translates at $\frac{d\underline{X}(t)}{dt} = \underline{U}(t)$ and rotates at $\underline{\Omega}(t)$ with respect to stationary reference frame $O123$.

$\underline{U}$ and $\underline{\Omega}$ (translation and rotation of non-inertial frame) can be resolved in either frame however time is invariant between both reference frames, i.e., $t' = t$.

Fluid particle location $P = P(\underline{x}')$ or $P(\underline{x})$ where $\underline{x}' = (x_1', x_2', x_3')$ and $\underline{x} = (x_1, x_2, x_3)$ and related by $\underline{x} = \underline{X} + \underline{x}'$

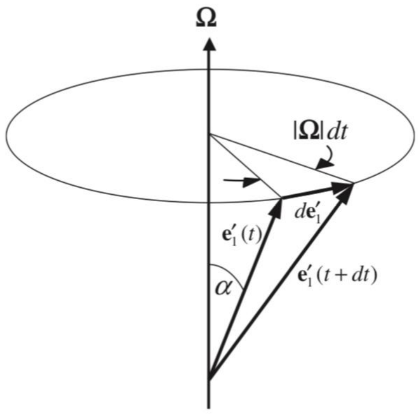
The velocity $\underline{u}$ (inertial frame) of P is:

$$\underline{u} = \frac{d\underline{x}}{dt} = \frac{d\underline{X}}{dt} + \frac{d\underline{x}'}{dt} = \underline{U} + \frac{d}{dt}(x_1'\boldsymbol{e_1'} + x_2'\boldsymbol{e_2'} + x_3'\boldsymbol{e_3'})$$

$$= \underline{U} + \underbrace{\frac{dx_1'}{dt}\boldsymbol{e_1'} + \frac{dx_2'}{dt}\boldsymbol{e_2'} + \frac{dx_3'}{dt}\boldsymbol{e_3'}}_{\underline{u'}}$$

$$+ \underbrace{x_1'\frac{d\boldsymbol{e_1'}}{dt} + x_2'\frac{d\boldsymbol{e_2'}}{dt} + x_3'\frac{d\boldsymbol{e_3'}}{dt}}_{\underline{\Omega} \times \underline{x'}} = \underline{U} + \underline{u'} + \underline{\Omega} \times \underline{x'}$$

The cross product $\underline{\Omega} \times \underline{x'} = x_1'\frac{d\boldsymbol{e_1'}}{dt} + x_2'\frac{d\boldsymbol{e_2'}}{dt} + x_3'\frac{d\boldsymbol{e_3'}}{dt}$ derivation based on geometric considerations. $\underline{u'} \neq \underline{u}$ due $\underline{U}$ and $\underline{\Omega}$.



$$\sin\alpha = \frac{r}{|\boldsymbol{e}_1'|} = r$$

$$r|\underline{\Omega}|dt = |d\boldsymbol{e}_1'| = \text{arc length}$$

$$(r\theta = s)$$

FIGURE 4.7 Geometry showing the relationship between $\boldsymbol{\Omega}$, the rotational velocity vector of O′1′2′3′, and the first coordinate unit vector $\mathbf{e}_1'$ in O′1′2′3′. Here, the increment $d\mathbf{e}_1'$ is perpendicular to $\boldsymbol{\Omega}$ and $\mathbf{e}_1'$.

Rotation $O'1'2'3'$per time increment dt causes, e.g., $\boldsymbol{e}_1'$ to trace a small portion of a cone with radius sinα.

$$|d\boldsymbol{e}_1'| = \sin\alpha\ |\underline{\Omega}|\ dt \Rightarrow \frac{|d\boldsymbol{e}_1'|}{dt} = \sin\alpha|\underline{\Omega}| = \underline{\Omega} \times \boldsymbol{e}_1' \quad \text{since } |\boldsymbol{e}_1'| = 1$$

Recognizing $\underline{\mathrm{a}} \times \underline{\mathrm{b}} = |\underline{\mathrm{a}}|\ |\underline{\mathrm{b}}| \sin\alpha\mathbf{n}$ where $\mathbf{n}$ is $\perp$ to both $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$,

i.e., $\frac{|d\boldsymbol{e}_1'|}{dt} = \underline{\Omega} \times \boldsymbol{e}_1'$.

$$\therefore\ \underline{\Omega} \times \underline{x'} = x_1' \frac{d\boldsymbol{e}_1'}{dt} + x_2' \frac{d\boldsymbol{e}_2'}{dt} + x_3' \frac{d\boldsymbol{e}_3'}{dt}$$

Acceleration:

$$\boldsymbol{a} = \frac{d\boldsymbol{u}}{dt} = \frac{d}{dt}(\boldsymbol{U} + \boldsymbol{u}' + \boldsymbol{\Omega} \times \boldsymbol{x}') = \frac{d\boldsymbol{U}}{dt} + \boldsymbol{a}' + 2\boldsymbol{\Omega} \times \boldsymbol{u}' + \frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{x}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}')$$

$\underline{U_t}$ = acceleration of O' w.r.t. O

$\boldsymbol{a}'$ = acceleration in O' reference frame

$2\underline{\Omega} \times \boldsymbol{u}'$ = Coriolis acceleration

$\frac{d\Omega}{dt} \times \boldsymbol{x}'$= acceleration in O′ due to $\underline{\dot{\Omega}}$

$\underline{\Omega} \times (\underline{\Omega} \times \boldsymbol{x}')$ = centripetal (inward) acceleration

For fluid: $\underline{a} = \frac{d\underline{u}}{dt} = \frac{D\underline{u}}{Dt}$ and $\underline{a'} = \frac{D\underline{u}'}{Dt}$, i.e., derivatives following the motion of the fluid particle.

$$\left(\frac{D\boldsymbol{u}}{Dt}\right)_{O123} = \left(\frac{D\boldsymbol{u}'}{Dt}\right)_{O'1'2'3'} + \frac{d\boldsymbol{U}}{dt} + 2\boldsymbol{\Omega} \times \boldsymbol{u}' + \frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{x}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}')$$

Substituting into the NS equation:

$$\rho\left(\frac{D\boldsymbol{u}'}{Dt}\right)_{O'1'2'3'} = -\nabla' p + \rho\left[\boldsymbol{g} - \underset{(1)}{\frac{d\boldsymbol{U}}{dt}} - \underset{(2)}{2\boldsymbol{\Omega} \times \boldsymbol{u}'} - \underset{(3)}{\frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{x}'} - \underset{(4)}{\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}')}\right] + \mu\nabla'^2\boldsymbol{u}'$$

provides the incompressible NS equations in non-inertial reference frame where ' denotes differentiation, velocity, and position in the O′ reference frame.

[ ] terms are body forces due $\boldsymbol{g}$ and extra terms due to the motion of O′.

Thermodynamic variables and net viscous stress are independent of reference frame such that $\nabla p = \nabla' p$ and $\mu\nabla^2\boldsymbol{u}' = \mu\nabla^2\boldsymbol{u}$. Appendix A

Recall $\nabla^2\boldsymbol{u} = 2\frac{\partial}{\partial x_j}\varepsilon_{ij} = \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{\partial^2 u_i}{\partial x_j \partial x_j}$.

For $\underline{U}$ = constant and $\underline{\Omega} = 0$: [ ] = $\underline{g}$ i.e. inertial reference frame.

For $\dot{\boldsymbol{U}}$ = constant, $\underline{\Omega}$ = constant, and $\boldsymbol{u}' = 0$: $[\,] = \underline{g} - \underline{\dot{U}} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}')$, i.e., rigid body translation and rotation such that $\mu\nabla^2\boldsymbol{u}' = 0$ and $\nabla p = \rho\left(\underline{g} - \underline{a}\right)$ with $\underline{a} = \underline{\dot{U}} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}')$.

(1) $\underline{\dot{U}}$ = acceleration O' relative to O

(2) $-2\underline{\Omega} \times \underline{u'}$ = Coriolis acceleration = $f(\boldsymbol{u'}) \neq f(\boldsymbol{x'})$

(3) $-\underline{\dot{\Omega}} \times \boldsymbol{x'} = f(\underline{\dot{\Omega}})$ i.e. rate of change of $\underline{\Omega}$

(4) $-\underline{\Omega} \times (\underline{\Omega} \times \boldsymbol{x'})$ = centrifugal acceleration = $f(\underline{\Omega}, \underline{x'})$

(1) Apparent force pushes person back into seat or tighten grip on handrail when vehicle accelerating. Aircraft parabolic trajectory weightless interior when $\dot{\boldsymbol{U}} = \underline{g}$.

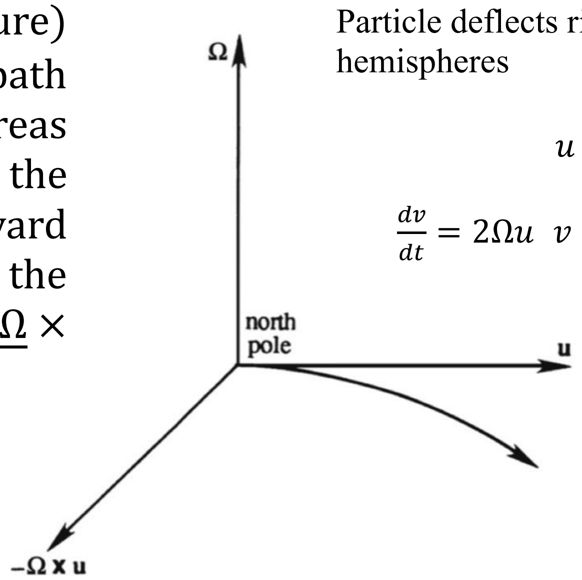
(2) $f(\underline{u'})$ not $\boldsymbol{x'}$. Important navigation (air/sea) and artillery.

$\underline{u'} = u\boldsymbol{i}$ $\underline{v'}$=0 $-\underline{\Omega} \times \underline{u'} = -\Omega u\boldsymbol{j} + \Omega v\boldsymbol{i} = -\Omega u\boldsymbol{j}$ acts perpendicular path therefore without change speed. Both $\boldsymbol{i}$ and $\boldsymbol{j}$ are towards the equator and **k** is upward (tangent plane coordinate system).

Ω = angular velocity earth, one cycle per day

Particle deflects right north & left south hemispheres

Here $\underline{u'}$ (**u** in figure) is in desired path direction, whereas the curved path is the actual rightward path due to the Coriolis force $-\underline{\Omega} \times \underline{u'}$.



$$u = \frac{dx}{dt} \quad x = ut$$

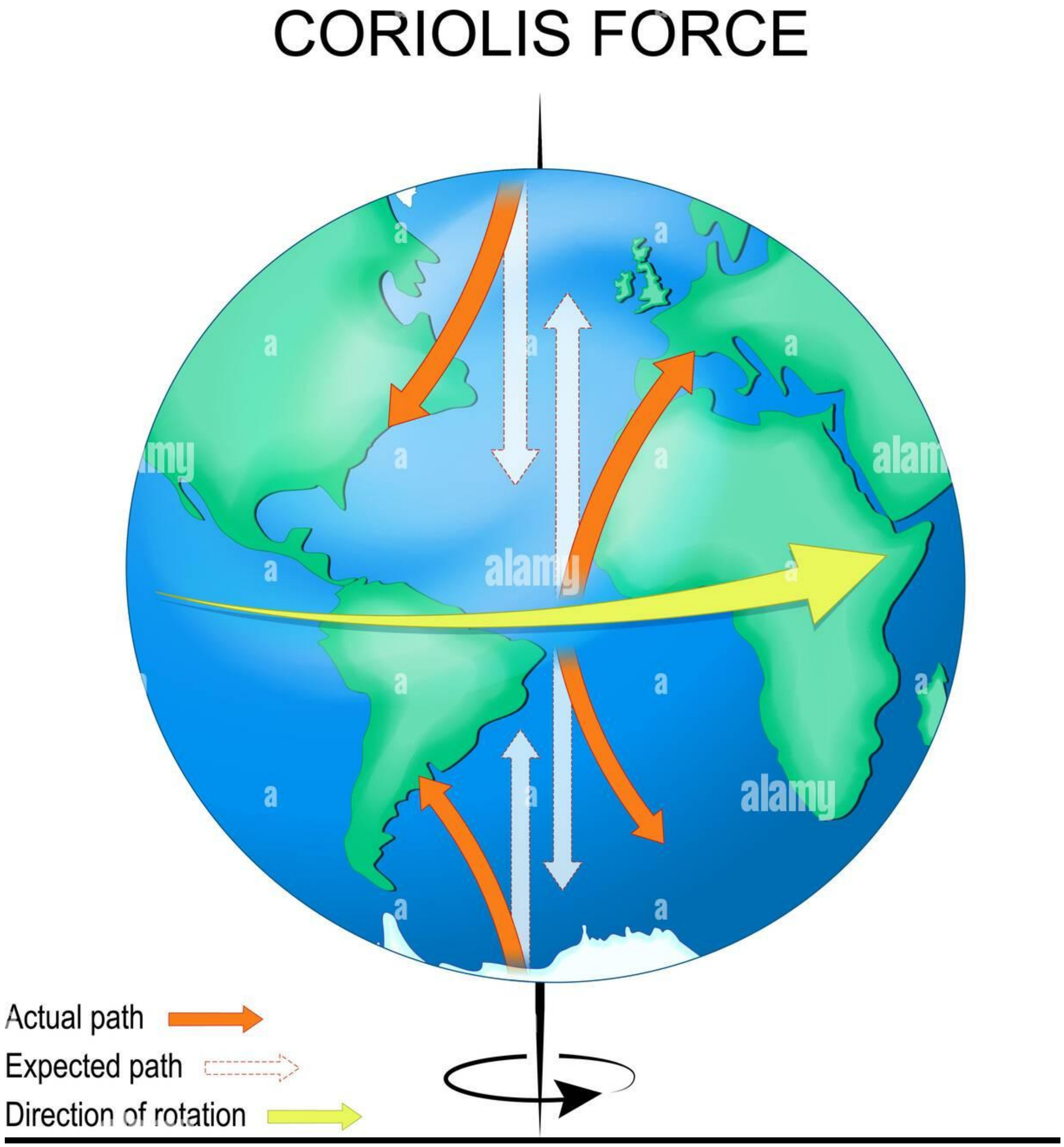
$$\frac{dv}{dt} = 2\Omega u \quad v = 2\Omega ut = \frac{dy}{dt} \quad y = \Omega ut^2$$

$$\tan\alpha = \frac{\Omega ut^2}{ut} = \Omega t$$

$\alpha = \tan^{-1}(\Omega t) \Rightarrow \alpha = \Omega t$

earth's rotation in time t

FIGURE 4.8 Particle trajectory deflection caused by the Coriolis acceleration when observed in a rotating frame of reference. If observed from a stationary frame of reference, the particle trajectory would be straight.



Expected = desired, whereas actual is "real" i.e., deflected path due Coriolis force. Therefore, must use leftward path to achieve desired path.

Also, important geophysical fluid dynamics.

(3) Important when $\underline{\Omega}(t)$ or direction of rotation changes with time

(4) Centrifugal acceleration = $f\left(\underline{\Omega}, \underline{x}'\right)$. $\underline{x}'$ = distance axis of rotation.

Consider: $\underline{\Omega} = (0,0,\Omega)$ $\quad x' = (R, \varphi, z)$

$$-\underline{\Omega} \times \left(\underline{\Omega} \times \underline{x}'\right) = \Omega^2 r\, \widehat{e_R} \text{ = apparent acceleration}$$

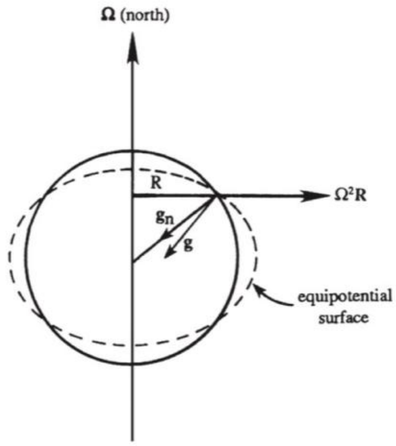
$$\underline{g} = \boldsymbol{g_n} + \Omega^2 r\, \widehat{e_R} \text{ = effective acceleration}$$

$\underline{g_n} = -\nabla\Phi$ ← Newtonian g_n towards earth center

$\Phi = gz$ Effective acceleration not towards earth center.

Body force potential can be found for new terms that are important in atmospheric or oceanic surface flows.

Earth surface ellipsoid with equatorial diameter 42 km longer than polar diameter



Varies over surface of earth

Perpendicular $\underline{g}$.

Average of sea level is equipotential surface

FIGURE 4.9 The earth's rotation causes it to budge near the equator and this leads to a mild distortion of equipotential surfaces from perfect spherical symmetry. The total gravitational acceleration is a sum of a centrally directed acceleration $\mathbf{g_n}$ (the Newtonian gravitation) and a rotational correction Ω^2R that points away from the axis of rotation.

$$\underline{u} = \frac{d\underline{x}}{dt} = \frac{d\underline{X}}{dt} + \frac{d\underline{x'}}{dt} = \underline{U} + \frac{d}{dt}(x_1'\boldsymbol{e_1'} + x_2'\boldsymbol{e_2'} + x_3'\boldsymbol{e_3'})$$

$$= \underline{U} + \underline{u'} + \underline{\Omega} \times \underline{x'}$$

$$\underline{u'} = \frac{dx_1'}{dt}\boldsymbol{e_1'} + \frac{dx_2'}{dt}\boldsymbol{e_2'} + \frac{dx_3'}{dt}\boldsymbol{e_3'}$$

$$\underline{\Omega} \times \underline{x'} = x_1'\dot{\boldsymbol{e}}_1' + x_2'\dot{\boldsymbol{e}}_2' + x_3'\dot{\boldsymbol{e}}_3'$$

$$\frac{d\underline{x'}}{dt} = \underline{u'} + \underline{\Omega} \times \underline{x'}$$

$$\frac{d\underline{u}}{dt} = \frac{d\underline{U}}{dt} + \frac{d\underline{u'}}{dt} + \frac{d}{dt}(\underline{\Omega} \times \underline{x'})$$

$$= \frac{d\underline{U}}{dt} + \frac{d}{dt}(u_1'\boldsymbol{e_1'} + u_2'\boldsymbol{e_2'} + u_3'\boldsymbol{e_3'}) + \frac{d\underline{\Omega}}{dt} \times \boldsymbol{x'} + \underline{\Omega} \times \frac{d\boldsymbol{x'}}{dt}$$

$$\frac{d\underline{u}}{dt} = \frac{d\underline{U}}{dt} + \underbrace{\left(\frac{du_1'}{dt}\boldsymbol{e_1'} + \frac{du_2'}{dt}\boldsymbol{e_2'} + \frac{du_3'}{dt}\boldsymbol{e_3'}\right)}_{\underline{a'}} + \underbrace{\left(u_1'\dot{\boldsymbol{e}}_1' + u_2'\dot{\boldsymbol{e}}_2' + u_3'\dot{\boldsymbol{e}}_3'\right)}_{\underline{\Omega}\times\underline{u'}} + \frac{d\underline{\Omega}}{dt} \times \underline{x'}$$
$$+ \underline{\Omega} \times (\underline{u'} + \underline{\Omega} \times \underline{x'})$$

$$= \frac{d\underline{U}}{dt} + \underline{a'} + 2\underline{\Omega} \times \underline{u'} + \frac{d\underline{\Omega}}{dt} \times \underline{x'} + \underline{\Omega} \times (\underline{\Omega} \times \underline{x'})$$

Inertial frame NS: $\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u}$

$$\therefore \frac{D\underline{u}}{Dt} = \frac{D\underline{u'}}{Dt} + \dot{\underline{U}} + 2\underline{\Omega} \times \underline{u'} + \frac{d\underline{\Omega}}{dt} \times \underline{x'} + \underline{\Omega} \times (\underline{\Omega} \times \underline{x'})$$
$$= -\frac{1}{\rho}\nabla p + \underline{g} + \nu \nabla^2 \underline{u'}$$

or $\rho \frac{Du'}{Dt} = -\nabla p + \rho\left[\underline{g} - \dot{\underline{U}} - 2\underline{\Omega} \times \underline{u'} + \frac{d\underline{\Omega}}{dt} \times \underline{x'} + \underline{\Omega} \times (\underline{\Omega} \times \underline{x'})\right] + \mu \nabla^2 u'$

Exercise 4.50. Derive (4.43) from (4.42).

Solution 4.50. The entire statement of equation (4.42) is:

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{X}}{dt} + \frac{d\mathbf{x}'}{dt} = \mathbf{U} + \frac{d}{dt}\left(x_1'\mathbf{e}_1' + x_2'\mathbf{e}_2' + x_3'\mathbf{e}_3'\right)$$

$$= \mathbf{U} + \frac{dx_1'}{dt}\mathbf{e}_1' + \frac{dx_2'}{dt}\mathbf{e}_2' + \frac{dx_3'}{dt}\mathbf{e}_3' + x_1'\frac{d\mathbf{e}_1'}{dt} + x_2'\frac{d\mathbf{e}_2'}{dt} + x_3'\frac{d\mathbf{e}_3'}{dt} = \mathbf{U} + \mathbf{u}' + \mathbf{\Omega}\times\mathbf{x}',$$

Thus, we see that $d\mathbf{x}'/dt = \mathbf{u}' + \mathbf{\Omega}\times\mathbf{x}'$. Now time differentiate $\mathbf{u}$, to find:

$$\frac{d\mathbf{u}}{dt} \equiv \mathbf{a} = \frac{d\mathbf{U}}{dt} + \frac{d\mathbf{u}'}{dt} + \frac{d}{dt}(\mathbf{\Omega}\times\mathbf{x}')$$

$$= \frac{d\mathbf{U}}{dt} + \frac{d}{dt}\left(u_1'\mathbf{e}_1' + u_2'\mathbf{e}_2' + u_3'\mathbf{e}_3'\right) + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{x}' + \mathbf{\Omega}\times\frac{d\mathbf{x}'}{dt}$$

$$= \frac{d\mathbf{U}}{dt} + \frac{du_1'}{dt}\mathbf{e}_1' + \frac{du_2'}{dt}\mathbf{e}_2' + \frac{du_3'}{dt}\mathbf{e}_3' + u_1'\frac{d\mathbf{e}_1'}{dt} + u_2'\frac{d\mathbf{e}_2'}{dt} + u_3'\frac{d\mathbf{e}_3'}{dt} + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{x}' + \mathbf{\Omega}\times\left(\mathbf{u}' + \mathbf{\Omega}\times\mathbf{x}'\right)$$

Here the various terms written in component form may be identified. The second through fourth terms are the fluid particle acceleration, $\mathbf{a}'$, observed in the non-inertial frame of reference. The fifth through seventh terms, which involve the time derivatives of the unit vectors, can be written in terms of a cross product:

$$u_1'\frac{d\mathbf{e}_1'}{dt} + u_2'\frac{d\mathbf{e}_2'}{dt} + u_3'\frac{d\mathbf{e}_3'}{dt} = \mathbf{\Omega}\times\mathbf{u}',$$

as depicted in Figure 4.7 and described in paragraph below (4.42). With these replacements, the last equality for the fluid particle acceleration becomes:

$$\mathbf{a} = \frac{d\mathbf{U}}{dt} + \mathbf{a}' + \mathbf{\Omega}\times\mathbf{u}' + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{x}' + \mathbf{\Omega}\times\mathbf{u}' + \mathbf{\Omega}\times\left(\mathbf{\Omega}\times\mathbf{x}'\right)$$

$$= \frac{d\mathbf{U}}{dt} + \mathbf{a}' + 2\mathbf{\Omega}\times\mathbf{u}' + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{x}' + \mathbf{\Omega}\times\left(\mathbf{\Omega}\times\mathbf{x}'\right),$$

which matches (4.43).

Right Circular Cone

Let R = radius of base, s = slant height.

$s = \sqrt{R^2 + h^2}$

$S = \pi R s = \pi R\sqrt{R^2 + h^2}$

$T = \pi R(R + s) = \pi R(R + \sqrt{R^2 + h^2})$

$V = \frac{1}{3}\pi R^2 h$