Chapter 3 Solutions of the Newtonian Viscous-Flow Equations

5. Unsteady flows

   a. Stokes 1st problem: sudden acceleration
   b. Diffusion vortex sheet
   c. Decay of a Line Vortex
   d. Burgers Vortex
   e. Stokes 2nd problem: steady oscillations
   f. Starting flow circular pipe
   g. Oscillating pressure gradient pipe flow
   h. Starting flow fixed/moving parallel walls.

a, e, and h are unsteady flows with moving boundaries, of which there are many additional solutions, including some that illustrate boundary layer behavior.

Stokes 1st Flat plate wall $y = 0$

$$u_t = \nu u_{yy}$$
$$u(y,0) = 0 \quad \text{IC}$$
$$u(0,t) = U \quad \text{impulsive start}$$
$$u(\infty,t) = 0 \quad \text{stagnant}$$

Diffusion Vortex Sheet Vortex Sheet $y = 0$

$$u_t = \nu u_{yy}$$
$$u(y,0) = U\,\mathrm{sgn}(y) \quad \text{IC jump condition}$$
$$\left.\begin{aligned} u(\infty,t) &= U \\ u(-\infty,t) &= -U \end{aligned}\right\} \quad \text{far field } \pm U$$

Decay ideal line vortex $r = 0$ suddenly stopped

$$u_{\theta_t} = \nu \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right]$$
$$u_\theta(r,0) = \frac{\Gamma}{2\pi r} \quad \text{IC potential vortex}$$
$$u_\theta(0,t) = 0 \quad \text{no slip}$$
$$u_\theta(\infty,t) = \frac{\Gamma}{2\pi r} \quad \text{outer potential vortex}$$

Line vortex imposed on fluid at rest $r = 0$ suddenly started

$$u_{\theta_t} = \nu \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right]$$
$$u_\theta(r,0) = 0 \quad u_\theta(0,0) = \infty$$
$$u_\theta(r,\infty) = \frac{\Gamma}{2\pi r}$$
$$u_\theta(\infty,t) = 0 \quad u_\theta(0,\mathrm{t}) = \infty$$

Stokes 2nd Flat plate wall $y = 0$

$$u_t = \nu u_{yy}$$
$$u(0,t) = U\cos\omega t$$
$$u(\infty,t) = 0$$

Unsteady pipe flow: $\rho u_t = -\widehat{p_x}(t) + \mu\left(u_{rr} + \frac{1}{r}u_r\right)$

Stokes 1st Problem: Impulsively Started Plate

Assume
ρ, μ constant
$u \neq f(x)$
$u = f(y, t)$

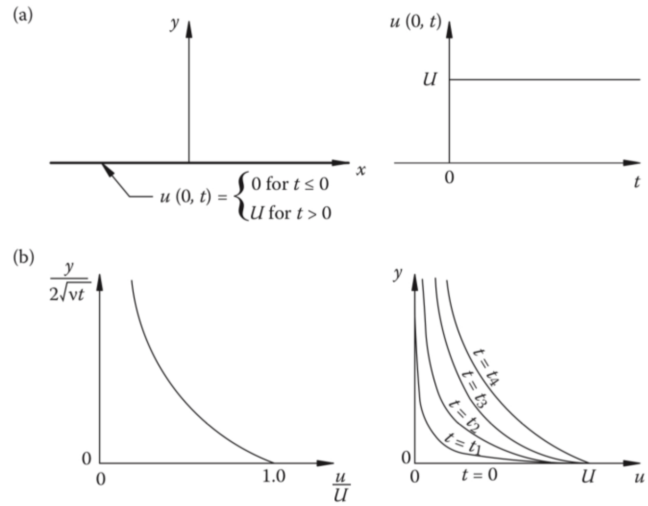


FIGURE 7.4
(a) Definition sketch for Stokes' first problem and (b) solution curves in terms of the similarity variable and in terms of dimensional variables.

(b) shows similarity (left) and u(t) for increasing t (right)

Continuity: $\begin{matrix} u_x + v_y = 0 \\ v(0) = 0 \end{matrix} \Rightarrow \quad v = 0$

Momentum: $$\rho u_t = -p_x + \mu u_{yy}$$
$$0 = -p_y$$

y momentum⇒ $p = p(x, t)$. For t≤0, u = 0 ∴ p = constant. For t≥0, u(∞, t) = 0 ∴ $p_x(\infty, t) = 0$ and in view y momentum equation $p_x(y, t) = 0$.

$\therefore u_t = \nu u_{yy}$
$u(y, 0) = 0$ IC
$\left. \begin{matrix} u(0, t) = U \\ u(\infty, t) = 0 \end{matrix} \right\}$ BC

Well posed IBVP

$$u = f(U, y, t, \nu)$$

Dimensional analysis: $\frac{u}{U} = F(\eta)$

η = nondimensional distance = $\frac{y}{2\sqrt{\nu t}}$ (2 for algebraic convenience)

Dimensionally reduced from 2 (y, t) to 1 (η) and similarity solution i.e. PDE → ODE

$$u_t = UF_t = UF'\eta_t = -UF'\frac{\eta}{2t}, \quad F' = F_\eta$$

$$u_y = UF_y = UF'\frac{1}{2\sqrt{\nu t}}$$

$$u_{yy} = \frac{U}{2\sqrt{\nu t}}F''\eta_y = \frac{U}{4\nu t}F''$$

$$\eta = \frac{y}{2\sqrt{\nu}}t^{-1/2}$$

$$\eta_t = \frac{y}{2\sqrt{\nu}}\left(-\frac{1}{2}t^{-3/2}\right)$$

$$= -\frac{y}{4\sqrt{\nu}}t^{-3/2}$$

$$= -\frac{\eta}{2t}$$

$$-2\eta\, d\eta = \frac{d\left(\frac{dF}{d\eta}\right)}{\frac{dF}{d\eta}}$$

$$-2\eta F' = F'' \quad F(\infty) = 0 \quad F(0) = 1$$

$u(y, 0) = 0$

$u(\infty, t) = 0$

$u(0, t) = U$

$$\frac{dF'}{F'} = -2\eta\, d\,\eta$$

$$\ln F' = -\eta^2 + \text{constant}$$

$$F' = Ae^{-\eta^2} \qquad e^{x+y} = e^x e^y$$

$$F(\eta) = A\int_0^\eta e^{-\eta^2} d\eta + B$$

$$F(0) = 1 = A\int_0^0 e^{-\eta^2} d\eta + B$$
$$B = 1$$

$$F(\infty) = 0 = A\int_0^\infty e^{-\eta^2} d\eta + 1$$

$$= A\frac{\sqrt{\pi}}{2} + 1$$
$$A = -\frac{2}{\sqrt{\pi}}$$

$$F = 1 - \underbrace{\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2} d\eta}_{erf(\eta)}$$

$$\frac{u}{U} = 1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)$$

$\omega(0,0) = \infty$ Dirac delta function vortex sheet
$\omega(y, 0) = 0$ due to impulsive motion

$\int_0^\infty \omega \, dy \neq f(t)$ i.e. no new ω after $t = 0$
Initial ω diffuses outward increasing flow width

$$\frac{u}{U} = 0.05 \Rightarrow \eta = 1.38$$

$\delta = 2.76\sqrt{\nu t}$ width of diffusion layer, which increases as $\sqrt{t}$

$\omega_z = -\frac{\partial u}{\partial y} = \frac{U}{2\sqrt{\nu t}} F' = \frac{U}{\sqrt{\pi \nu t}} e^{-\eta^2}$

$\tau_{yx} = \mu \frac{\partial u}{\partial y} = -\mu \omega_z$

$\frac{\tau_{yx}\sqrt{\pi \nu t}}{\mu U} = -e^{-\eta^2}$ nondimensional shear stress $\mu = \frac{Ns}{m^2}$ and $\nu = \frac{m}{s^2}$

$= f(\eta)$ only

$\tau_{yx}(0) = -\frac{\mu U}{\sqrt{\pi \nu t}} = \infty$ at $t = 0$

$\propto \frac{1}{\sqrt{t}} \quad t > 0$

Note: erf(η) ~ $\eta^{-1} e^{-\eta^2}$ as η to ∞

∴ influence plate extends to ∞ albeit exponentially small

$$\eta\left(\frac{u}{U} = 0.01\right) = 1.8 = \frac{\delta}{2\sqrt{\nu t}}$$

$$\delta\left(\frac{u}{U} = 0.01\right) = 3.6\sqrt{\nu t}$$

diffusion distance, i.e., effects of μ/ν within δ and independent of U

for $t = 1$ min air = 10.8 cm water = $2.8\ cm$

$\nu_{\text{air}} = 0.15\,\text{cm}^2/\text{s}$

$\nu_{\text{water}} = 0.01\,\text{cm}^2/\text{s}$

Stokes (1851)

Rayleigh (1881) skin friction law: laminar flow flat plate L, $U_0, t = \frac{x}{U_0}$

$\tau = \frac{\mu U_0}{\sqrt{\pi \nu}} \sqrt{\frac{U_0}{x}} = \frac{\mu U_0^{3/2}}{\sqrt{\pi \nu}} x^{-1/2} = 0.56\,\mu U_0^{3/2}/\sqrt{\nu x}$

$D = \int_0^L \tau_x\ dx = \frac{\mu U_0^{3/2}}{\sqrt{\pi \nu}}\ 2L^{1/2}$ per unit span and $C_D = \frac{D}{\frac{1}{2}\rho U^2 L} = \frac{2.26}{\sqrt{Re_L}}$

vs. Blasius $\tau_W = \mu \frac{U f''(0)}{\sqrt{2\nu x/U}} = \frac{\mu U^{3/2} 0.5}{\sqrt{2\nu}} x^{-1/2} = 0.35 \frac{\mu U^{3/2}}{\sqrt{\nu x}}$ for which

$$C_D = \frac{1.328}{\sqrt{Re_L}}$$

$$\omega = u_y = \frac{U}{\sqrt{\pi \nu t}} e^{-\eta^2} \qquad \eta = \frac{y}{2\sqrt{\nu t}}$$

$$\omega(0,0) = \frac{1}{0} e^0 = \frac{1}{0} = \infty \quad \eta^2 = \frac{y^2}{4\nu t}$$

$$\eta^2(0,0) = \frac{0}{0} = \frac{2y}{4\nu} = 0 \quad (y = 0)$$

$$\omega(y,0) = \frac{U}{0} e^{-\frac{y^2}{0}} = \frac{0}{0}$$

$$= \frac{U e^{-\eta^2}(-2\eta)}{\sqrt{\pi\nu}\left(\frac{1}{2}t^{-1/2}\right)} \qquad \eta = \frac{y}{0} = \infty \quad \eta^2 = \frac{y^2}{0} = \infty$$

$$= \frac{U e^{-\eta^2}(-2\eta) 2\sqrt{\nu}}{\sqrt{\pi\nu}} = U e^{-\eta^2} \left(\frac{-2y}{2\sqrt{\nu t}}\right) 2\sqrt{\nu} = 0$$

Initially ω(0,0) infinite and for $y > 0$ $\omega(y,0) = 0$.

ω = Gaussian with width increase $\sqrt{t}$ (as per δ) and max value decrease $\frac{1}{\sqrt{t}}$.

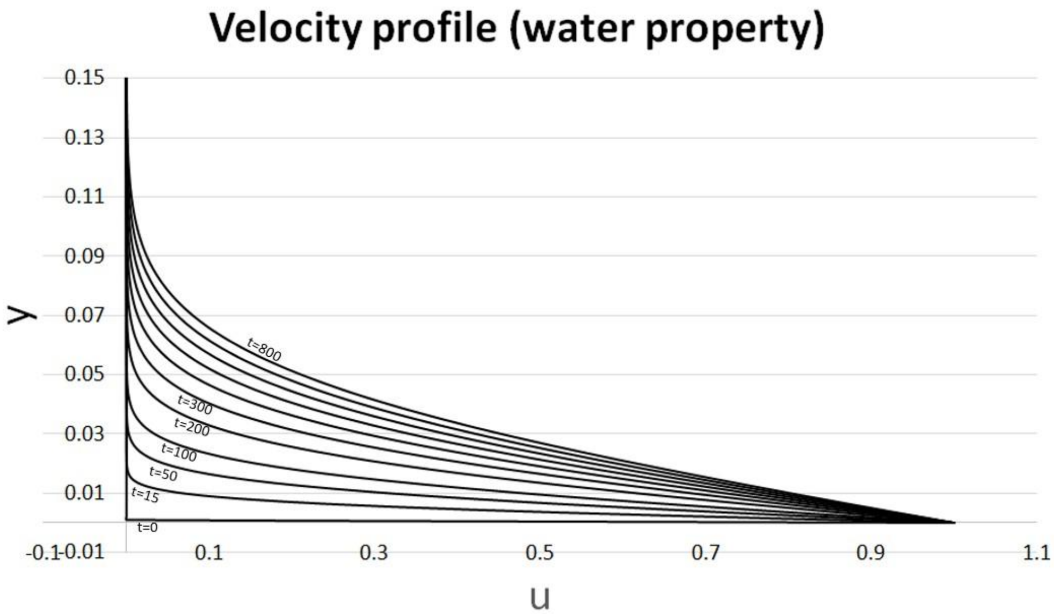
Total amount vorticity

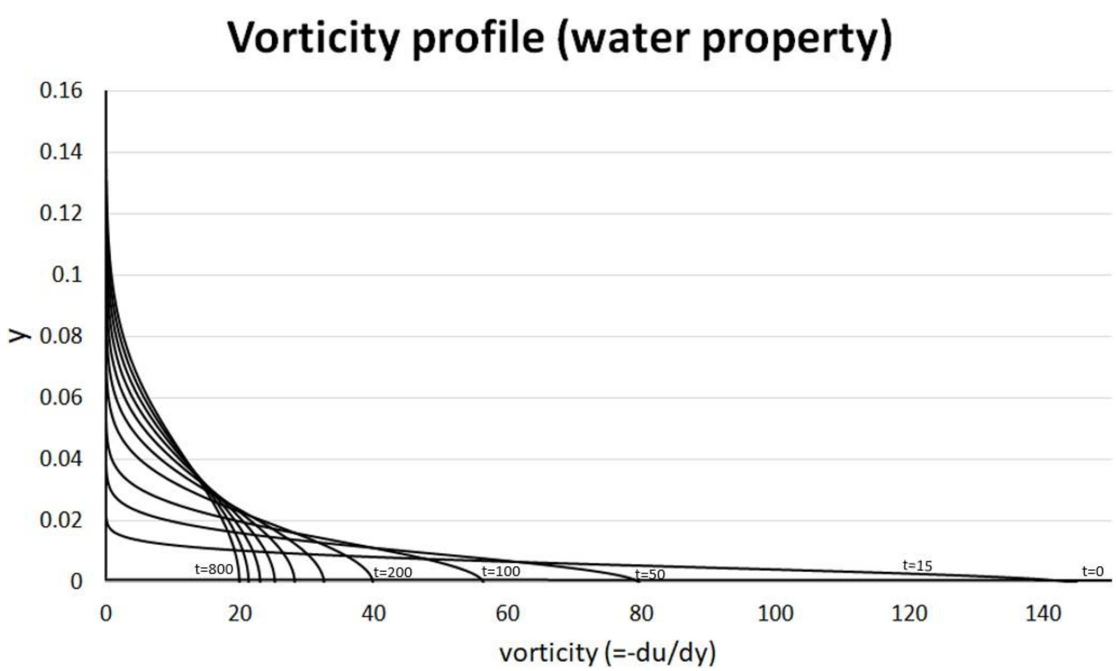
$$\int_0^\infty \omega \, dy = 2\sqrt{\nu t} \int_0^\infty \omega \, d\eta = \frac{U}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta = U \quad \neq f(t)$$

= y integral initial Dirac delta function ω. Gaussian integral = $\sqrt{\pi}$. ∴ no new vorticity created after t = 0.

Initial vorticity simply diffuses outward resulting in increased flow width.

$$\mu: 0.001 kg/m\ s$$
$$\rho: 1000\text{kg}/m^3$$
$$\nu: 1 \times 10^{-6} m^2/s$$
$$U = 1\text{m/s}$$
$$\frac{\boldsymbol{u}}{\boldsymbol{U}} = \mathbf{1} - \boldsymbol{erf}\left(\frac{\boldsymbol{y}}{\mathbf{2}\sqrt{\boldsymbol{\nu t}}}\right)$$



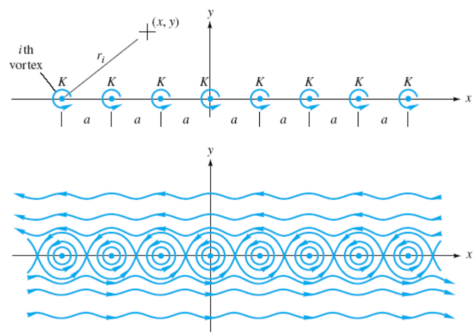


Diffusion vortex sheet

First recall potential flow solution for vortex sheet. The point vortex singularity is important in aerodynamics; since, their distributions can be used to represent airfoils and wings. To see this, consider as an example of an infinite row of vortices:

$$\psi = -K\sum_{i=1}^{\infty}\ln r_i = -\frac{1}{2}K\ln\left[\frac{1}{2}(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a})\right]$$

Where r_i is radius from origin of ith vortex.



Superposition infinite row equally spaced vortices of equal strength

For $|y| \geq a$ the flow approaches uniform flow with

$$u = \frac{\partial\psi}{\partial y} = \pm\frac{\pi K}{a}$$

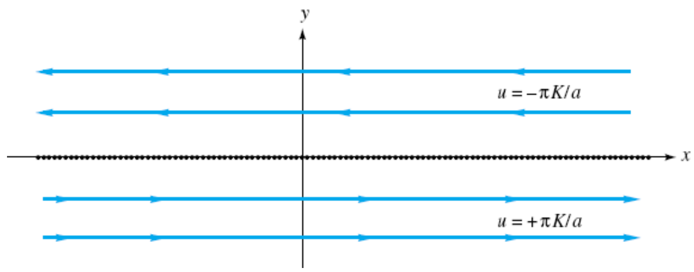
+: below x axis

-: above x axis

Note: this flow is just due to infinite row of vortices and there isn’t any pure uniform flow

Potential Flow Vortex sheet:

From afar (i.e. $|y| \geq a$) looks like a thin sheet with velocity discontinuity.



Define $\gamma = \dfrac{2\pi K}{a} =$ strength of vortex sheet

$d\Gamma = \underline{V} \cdot d\,\underline{s}$ (around closed contour)

$$d\Gamma = u_l dx - u_u dx = (u_l - u_u)dx = \frac{2\pi K}{a} dx$$

i.e. $\gamma = \dfrac{d\Gamma}{dx}$ = Circulation per unit span

Note: There is no flow normal to the sheet so that vortex sheet can be used to simulate a body surface. This is the basis of airfoil theory where we let $\gamma = \gamma(x)$ to represent body geometry.

Diffusion of a Vortex Sheet

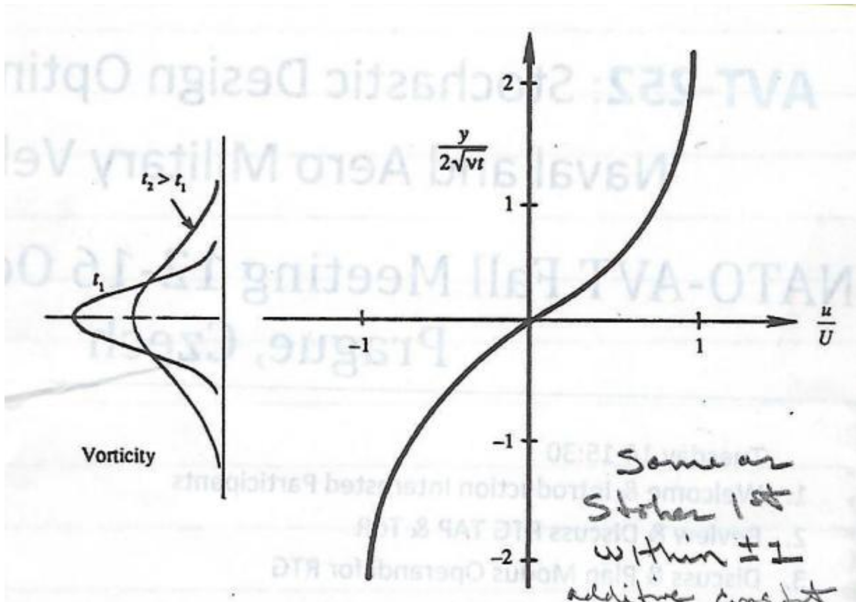


Figure 9.11 Viscous decay of a vortex sheet. The right panel shows the nondimensional solution and the left panel indicates the vorticity distribution at two times.

$$u_t = \nu u_{yy}$$
$$u(y,0) = U\,\mathrm{sgn}(y)$$
$$u(\infty,t) = U$$
$$u(-\infty,t) = -U$$

$$\frac{u}{U} = F(\eta) \qquad \eta = \frac{y}{2\sqrt{\nu t}}$$

Same as Stokes 1st

Initial ω diffuses away from y = 0

$$\omega = -\frac{\partial u}{\partial y} = -\frac{U}{\sqrt{\pi \nu t}} e^{-y^2/4\nu t}$$

Gaussian with width increase $\sqrt{t}$

$\int_{-\infty}^{\infty} \omega \; dy = 2\sqrt{\nu t} \int_{-\infty}^{\infty} \omega \; d\eta = \frac{2U}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta = 2U \neq f(t)$ and y integral initial ω = Dirac delta function

$$F'' = -2\eta F'$$
$$F(\infty) = 1$$
$$F(-\infty) = -1$$

$$F(\eta) = \mathrm{erf}(\eta)$$

$$u = U\mathrm{erf}(\eta)$$

Note: $\frac{d(\mathrm{erf}(y))}{dy} = \frac{2}{\sqrt{\pi}} e^{-y^2}$

$$u = \pm 0.95U$$
$$\eta = \pm 1.38$$
$$\delta = \pm 5.52\sqrt{\nu t}$$

The diffusion of a vortex sheet is related to both Stokes 1st and laminar flat plate boundary layer for several reasons.

1. u/U upper half = Stokes 1st after Galilean transformation (relative inertial coordinates) to coordinate system moving at speed U including sign change.

2. Flow y > 0 = temporally developing BL at t = 0. For large y vorticity is zero and u = U and no slip is satisfied a y = 0.

$$\tau_w = \mu u_y|_{y=0} = \frac{\mu U}{\sqrt{\pi \nu t}} \quad C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{\nu}{U^2 t}}$$

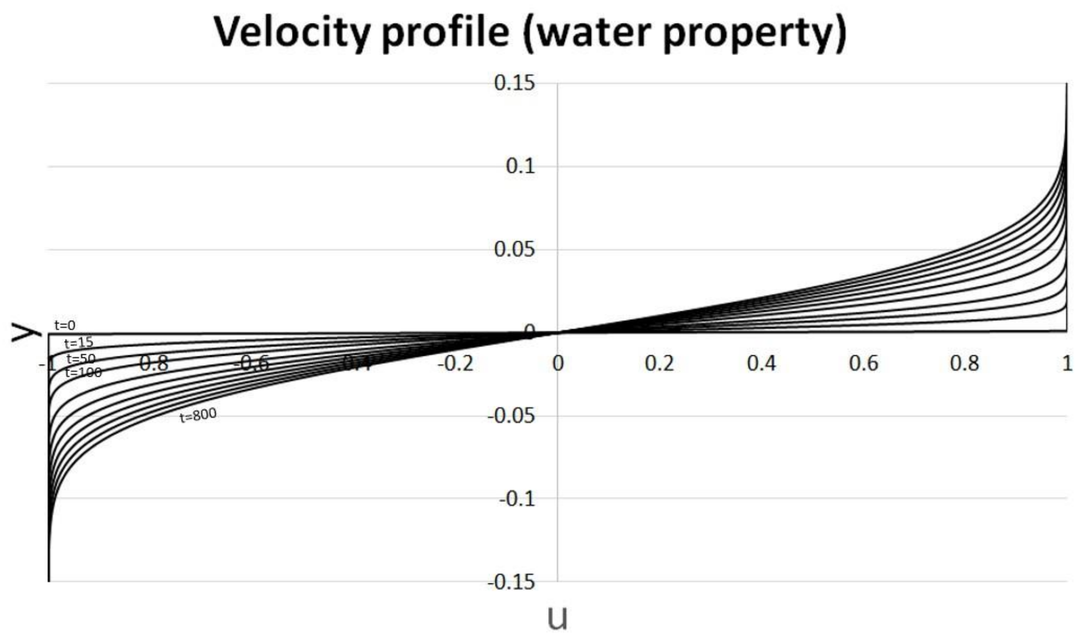
which are f(t).

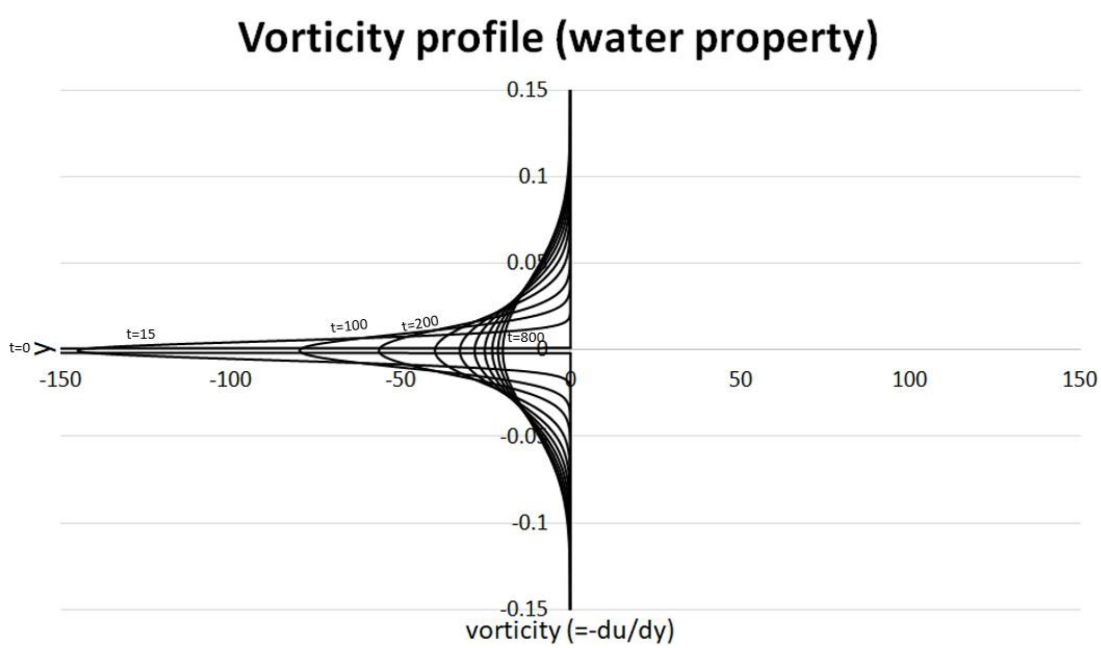
In fluid dynamics, a **temporally developing boundary layer** (TBL) refers to a flow configuration where the boundary layer's growth is analyzed as a function of **time** rather than streamwise distance. This is often modeled as the turbulent counterpart to the **Rayleigh problem** (or Stokes' first problem), where a fluid at rest is set into motion by an impulsively started or constant-velocity wall.

3. Ut = x transforms solution to spatially developing BL, as per laminar BL theory.

$$\sqrt{\frac{\nu}{U^2 t}} = \left(\frac{\nu}{Ux}\right)^{1/2} = Re_x^{-1/2}$$

$$\mu: 0.001 kg/m\ s$$
$$\rho: 1000 kg/m^3$$
$$\nu: 1 \times 10^{-6} m^2/s$$
$$U = 1\text{m/s}$$
$$\boldsymbol{u} = \boldsymbol{Uerf}\left(\frac{\boldsymbol{y}}{\boldsymbol{2\sqrt{\nu t}}}\right)$$





Decay of an ideal line vortex: Oseen Vortex

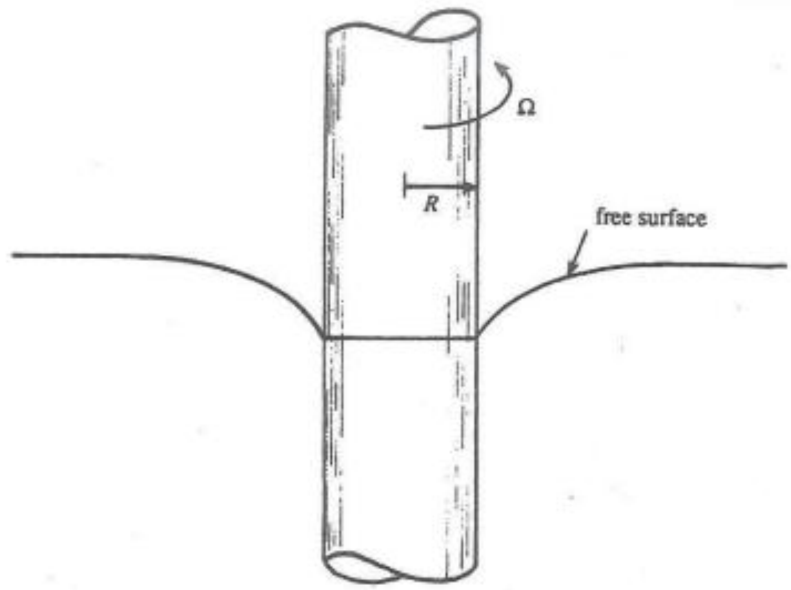


Figure 9.7 Rotation of a solid cylinder of radius R in an infinite body of viscous fluid. The shape of the free surface is also indicated. The flow field is viscous but irrotational.

$$u_\theta = \frac{\Omega R^2}{r} = \frac{\Gamma}{2\pi r} \quad \Gamma = 2\pi\,\Omega\,R^2$$

Irrotational vortex $\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] = -\frac{2\mu\,\Omega\,R^2}{r^2}$

work done per unit height $= 2\pi R\,\tau_{r\theta}\,u_\theta|_{r=R}$

$= 2\pi R(-2\mu\,\Omega\,)(\,\Omega\,R) = 4\pi\mu R^2\,\Omega^{\,2}$

= viscous dissipation such that there is no net force at a point

Suppose $r \to 0$ while $\Omega \uparrow$ such that $\Gamma = 2\pi\,\Omega\,R^2$ is unchanged. In the limit we have a potential line vortex with singularity at the origin.

Exercise 9.15. Consider a solid cylinder of radius a, steadily rotating at angular speed Ω in an infinite viscous fluid. The steady solution is irrotational: $u_\theta = \Omega a^2/R$. Show that the work done by the external agent in maintaining the flow (namely, the value of $2\pi R u_\theta \tau_{r\theta}$ at $R = a$) equals the viscous dissipation rate of fluid kinetic energy in the flow field.

Solution 9.15. Using the given velocity field, the shear stress is:

$$\tau_{R\varphi} = \mu R \frac{\partial}{\partial R}\left(\frac{u_\varphi}{R}\right) = \mu \Omega a^2 R \frac{\partial}{\partial R}\left(\frac{1}{R^2}\right) = -2\mu\Omega a^2 \frac{1}{R^2}.$$

The work done per unit height = $\left\{2\pi a \tau_{R\varphi} u_\varphi\right\}_{R=a} = 2\pi a \cdot 2\mu\Omega \cdot \Omega a = 4\pi\mu a^2\Omega^2$.

From (4.58) the viscous dissipation rate of kinetic energy per unit volume for an incompressible flow is $\rho\varepsilon = 2\mu S_{ij}S_{ij}$, where ε is the viscous dissipation of kinetic energy per unit mass. For the given flow field there is only one non-zero independent strain component:

$$S_{R\varphi} = S_{\varphi R} = \frac{R}{2}\frac{\partial}{\partial R}\left(\frac{u_\varphi}{R}\right) = \frac{\Omega a^2}{2} R \frac{\partial}{\partial R}\left(\frac{1}{R^2}\right) = -\Omega a^2 \frac{1}{R^2}.$$

Therefore:

$$\rho\varepsilon = 2\mu S_{ij}S_{ij} = 2\mu\left(S_{R\varphi}^2 + S_{\varphi R}^2\right) = 4\mu\Omega^2 \frac{a^4}{R^4},$$

so the kinetic energy dissipation rate per unit height is:

$$\int_a^\infty \rho\varepsilon 2\pi R dR = 8\pi\mu\Omega^2 a^4 \int_a^\infty \frac{1}{R^3} dR = 4\pi\mu\Omega^2 a^2,$$

which equals the work done turning the cylinder.

$$\frac{\partial}{\partial x_j}(U_i \sigma_{ij}) = \sigma_{ij}\frac{\partial U_i}{\partial x_j} + U_i \frac{\partial \sigma_{ij}}{\partial x_j}$$

$\frac{\partial}{\partial x_j}(U_i \sigma_{ij})$	$\sigma_{ij}\frac{\partial U_i}{\partial x_j}$	$U_i \frac{\partial \sigma_{ij}}{\partial x_j}$
Total work of surface force	Deformation work w/o $\underline{a}$ and lost to internal energy	Increase of KE since contributes fluid $\underline{a}$

$$\sigma_{ij}\frac{\partial U_i}{\partial x_j} = \sigma_{ij}\left(\varepsilon_{ij} + \omega_{ij}\right) = \sigma_{ij}\varepsilon_{ij}$$

$\sigma_{ij}\omega_{ij} = 0$ since it is the product of a symmetric and an anti-symmetric tensor.

$$\sigma_{ij}\frac{\partial U_i}{\partial x_j} = \left[-\left(p + \frac{2}{3}\mu\nabla\cdot\underline{U}\right)\delta_{ij} + 2\mu\varepsilon_{ij}\right]\varepsilon_{ij}$$

$$\sigma_{ij}\frac{\partial U_i}{\partial x_j} = -p\nabla\cdot\underline{U} + \underbrace{2\mu\varepsilon_{ij}\varepsilon_{ij} - \frac{2}{3}\mu\left(\nabla\cdot\underline{U}\right)^2}_{\varphi}$$

Since $\varepsilon_{ij}\delta_{ij} = \varepsilon_{ii} = \nabla\cdot\underline{U}$

$$\sigma_{ij}\frac{\partial U_i}{\partial x_j} = -p\nabla\cdot\underline{U} + \varphi$$

Next, suppose infinitely small R and rapidly spinning Ω cylinder suddenly stops rotating at t = 0; thereby, reducing the velocity at r = 0 to zero impulsively. Then the fluid would slow down due to viscous diffusion, i.e., viscous decay of a line vortex. Circular analog of the diffusion of a vortex sheet.

$$u_{\theta_t} = \nu \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right]$$
$$u_\theta(r,0) = \frac{\Gamma}{2\pi r}$$
$$u_\theta(0,t) = 0$$
$$u_\theta(\infty,t) = \frac{\Gamma}{2\pi r}$$

Assume $u' = \frac{u_\theta}{\Gamma/2\pi r} = \underbrace{f(r,t,\nu)}_{\text{must be nondimensional}} = F(\eta) \qquad \eta = \frac{r^2}{4\nu t}$

Expect similarity since r and t have no natural scales from BC; and eliminate $\Gamma/2\pi r$ via nondimensional u_θ. Note η is square of form used for Stokes 1st problem. Substitute u' into GDE.

$$F'' + F' = 0$$
$$F(\infty) = 1$$
$$F(0) = 0$$

$$u_\theta = \frac{\Gamma}{2\pi r} u' = \frac{\Gamma}{2\pi r} F(\eta) \qquad \eta = \frac{r^2}{4\nu t} = \frac{r^2}{4\nu} t^{-1}$$

$$\eta_t = -\frac{r^2}{4\nu} t^{-2}$$

$$\eta_r = \frac{2r}{4\nu t} = \frac{r}{2\nu t}$$

$$u_{\theta_t} = \frac{\Gamma}{2\pi r} F' \eta_t = -\frac{\Gamma r}{8\pi\nu t^2} F'$$

$$r u_\theta = \frac{\Gamma}{2\pi} F \qquad \frac{\partial}{\partial r}(r u_\theta) = \frac{\Gamma}{2\pi} F' \frac{r}{2\nu t} = \frac{\Gamma r}{4\pi\nu t} F' \div r = \frac{\Gamma}{4\pi\nu t} F'$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right] = \frac{\Gamma}{4\pi\nu t} F'' \frac{r}{2\nu t} = \frac{\Gamma r}{8\pi\nu^2 t^2} F''$$

$$-\frac{\Gamma r}{8\pi\nu t^2} F' = \frac{\Gamma r}{8\pi\nu t^2} F'' \quad \text{i.e.} \quad F'' + F' = 0$$

$$\frac{F''}{F'} = -1 \qquad \frac{1}{F'}\frac{d}{d\eta}(F') = -1 \qquad \frac{dF'}{F'} = -d\,\eta$$

$$\ln F' = -\eta + C$$

$$F(\infty) = 1 \qquad F' = C e^{-\eta}$$

$$F(0) = 0 \qquad F = C e^{-\eta} + D$$

$$F = 1 - e^{-\eta}$$

D = 1, C+D=0, C=−D

$$u_\theta = \frac{\Gamma}{2\pi r}\left[1 - e^{-r^2/4\nu t}\right]$$

$$F = 1 - e^{-\eta}$$

$$u_\theta = \frac{\Gamma}{2\pi r}\left[1 - e^{-r^2/4\nu t}\right]$$

Different form $\eta = \frac{r}{\sqrt{\nu t}}$ $u_\theta = \frac{\Gamma}{2\pi\sqrt{\nu t}}\left(\frac{1}{\eta}\right)\left[1 - \exp\left(-\frac{\eta^2}{4}\right)\right]$

For which former $\frac{u_\theta}{r}$ = f(η) whereas latter $\frac{u_\theta}{\sqrt{t}}$ = f(η).

One of family of vortices that satisfy NS equations. Another is the Taylor vortex.

$$u_\theta = \frac{H}{8\pi\nu t}\frac{r}{\nu t^2}\exp\left(-\frac{r^2}{4\nu t}\right)$$

H = amount of angular momentum in the vortex, which is infinite for Oseen vortex.

Oseen
$V^* \propto \frac{u_\theta}{\sqrt{t}}$

Taylor
$V^* \propto \frac{u_\theta}{t^{-3/2}}$

$\eta = \frac{r}{\sqrt{\nu t}}$

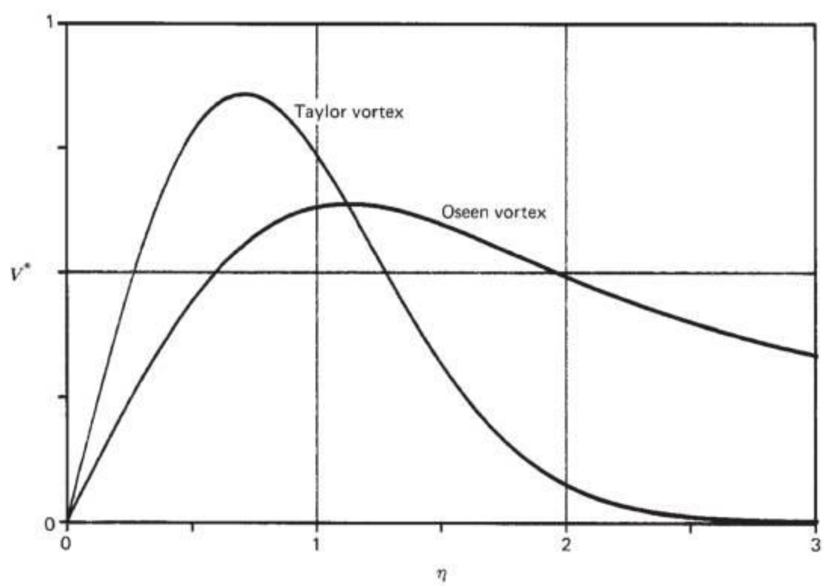


Figure 11.8 Profiles for Oseen and Taylor vortices in similarity variables. For the Oseen vortex, $V^* \propto v_\theta/t^{-1/2}$, while for the Taylor vortex, $V^* \propto v_\theta/t^{-3/2}$. In each case $\eta = r/\sqrt{\nu t}$.

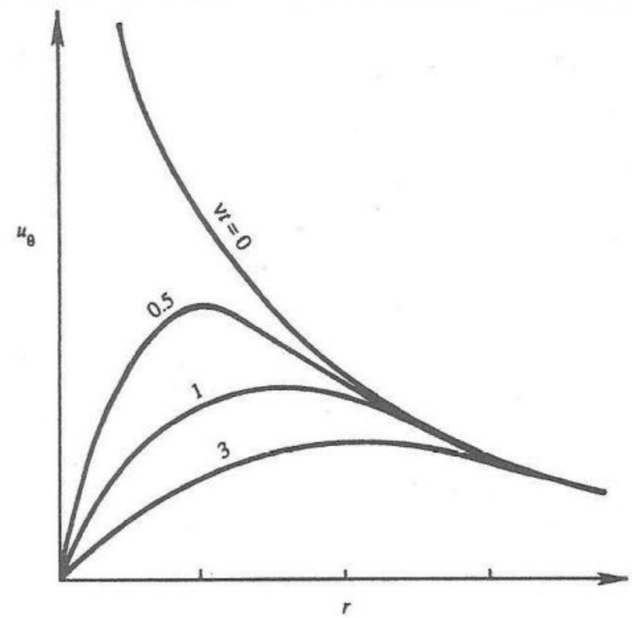


Figure 9.12 Viscous decay of a line vortex showing the tangential velocity at different times.

For $r \ll 2\sqrt{\nu t}$ rigid body rotation

$r \gg 2\sqrt{\nu t}$ irrotational vortex

Alternatively, a corresponding solution can be obtained for a line vortex suddenly imposed on a fluid at rest. Impulsive start of infinitely small R fast Ω cylinder. In this case,

$$u_\theta = \frac{\Gamma}{2\pi r} e^{-r^2/4\nu t}$$

For the Oseen vortex,

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) = \frac{\Gamma}{4\pi\nu t}\exp\left(-\frac{\eta^2}{4}\right) \qquad \eta = \frac{r}{\sqrt{\nu t}}$$

i.e., Gaussian bell curve profile at each instant.

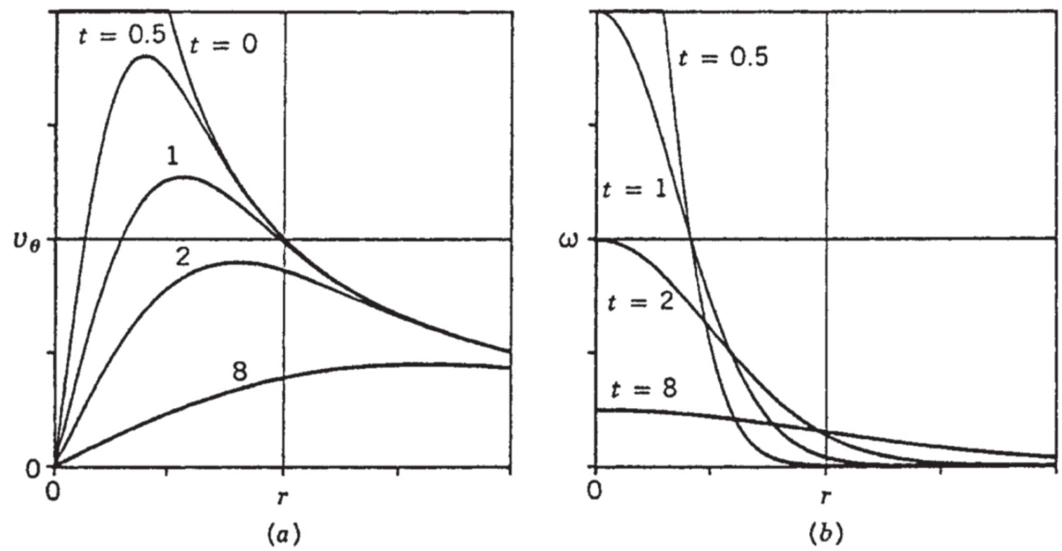


Figure 11.7 Viscous decay of an ideal vortex: (*a*) velocity profiles and (*b*) vorticity profiles at corresponding times. Scales are arbitrary.

When $\omega_z \neq 0$ flow is viscous, whereas when $\omega_z = 0$ it remains the original potential flow vortex. Height falls off as t^{-1} and width increases by viscous diffusion $\sqrt{\nu t}$.

Note: $\int_0^\infty \omega_z \; 2\pi r\, dr = \;\Gamma$

Solution useful estimate decay of wing/propeller tip vortices, especially if eddy viscosity ν_t used.

Squire (1965)

$\nu_t \;= f\left(Re_{\text{vortex}} = \;\Gamma\,/\nu = \dfrac{2\pi r u_\theta}{\nu}\right)$

Decay time $t = \frac{z - z_0}{U}$ $\;\Delta\, z = z - z_0$ distance behind wing

$u_{\theta,\max\, decay} \propto z^{-1/2}$ U = aircraft flight speed

Core growth $\propto z^{1/2}$ $\;z_0$ = effective origin

Alternative derivation viscous decay line vortex

Similarity solutions:

$\gamma = At^{-n}F\big(\xi/\delta(t)\big) = At^{-n}F(\eta)$
or
$\gamma = A\xi^{-n}F\big(\xi/\delta(t)\big) = A\xi^{-n}F(\eta)$

γ = dependent field variable, e.g., velocity component
A = constant $\gamma \propto t^n$ or $\gamma \propto \xi^n$
ξ = independent spatial variable
t = time
$\eta = \xi/\delta$ = similarity variable
$\delta(t)$ = time dependent length scale
$At^{-n} or A\xi^{-n} \times F$ needed when solutions are infinite or zero at t = 0 or ξ = 0

Thin rapidly spinning cylinder $u_\theta = \frac{\Gamma}{2\pi r}$, i.e., ideal vortex strength Γ located at r = 0. At t = 0 cylinder stops spinning.

$$u_\theta(r,t) = Ar^{-n}F\left(\frac{r}{\delta(t)}\right) = Ar^{-n}F(\eta)$$
$$u_\theta(r,0) = \frac{\Gamma}{2\pi r} = u_\theta(r \to \infty, t)$$
$$u_\theta(0,t) = 0 \quad t > 0$$

i.e. $F(\eta \to \infty) = 1$ and F(0) = 0, $Ar^{-n} = \frac{\Gamma}{2\pi r}$

$$\frac{\partial u_\theta}{\partial t} = \nu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right) \qquad u_\theta = \left(\frac{\Gamma}{2\pi r}\right)F(\eta) \qquad \eta = \frac{r}{\delta(t)} \qquad \eta_t = -\frac{r}{\delta^2}\delta_t$$

$$\frac{\partial u_\theta}{\partial t} = \frac{\Gamma}{2\pi r}F'\left(-\frac{r}{\delta^2}\right)\delta_t = -\frac{\Gamma}{2\pi r}(\delta^{-1}\delta_t)\,\eta F'$$

$$ru_\theta = \frac{\Gamma}{2\pi}F \qquad \frac{\partial}{\partial r}(ru_\theta) = u_\theta + \frac{\Gamma}{2\pi}\left(\frac{F'}{\delta} - \frac{F}{r}\right)$$

$$\frac{\partial u_\theta}{\partial r} = -\frac{\Gamma}{2\pi r^2}F + \frac{\Gamma}{2\pi r}F'\delta^{-1} = \frac{\Gamma}{2\pi r}\left(\frac{F'}{\delta} - \frac{F}{r}\right)$$

$$\frac{\partial}{\partial r}\left(\frac{u_\theta}{r} + \frac{\Gamma}{2\pi r}\left(\frac{F'}{\delta} - \frac{F}{r}\right)\right)$$

$$\frac{\partial}{\partial r}\left(\cancel{\frac{\Gamma}{2\pi r^2}F} + \frac{\Gamma}{2\pi r}\left(\frac{F'}{\delta} - \cancel{\frac{F}{r}}\right)\right)$$

$$\frac{\Gamma}{2\pi}\frac{\partial}{\partial r}\left(\frac{F'}{r\delta}\right) = \frac{\Gamma}{2\pi}\left[\frac{1}{r\delta^2}F'' - \frac{F'}{r^2\delta}\right]$$

$$-\frac{\cancel{\Gamma}}{\cancel{2\pi} r}(\delta^{-1}\delta_t)\eta F' = \frac{\nu\cancel{\Gamma}}{\cancel{2\pi}}\left(\frac{1}{r\delta^2}F'' - \frac{F'}{r^2\delta}\right)$$

For similarity $[\,] = f(\eta) \neq f(r,t)$

$\therefore$ assume $\delta = \sqrt{\nu t}$

$$\delta_t = \frac{\nu^{1/2}}{2}t^{-1/2}$$

$$\frac{r^2}{\nu\cancel{\nu^{1/2}}t^{1/2}}\frac{\cancel{\nu^{1/2}}}{2}t^{-1/2} = \frac{r^2}{2\nu t} = \frac{\eta^2}{2}$$

$$-\left[\frac{r^2}{\nu\delta}\frac{d\delta}{dt}\right]\eta F' = \frac{r^2}{\delta^2}F'' - \frac{r}{\delta}F'$$

$$-\frac{\eta^3}{2}F' = -\eta^2 F'' - \eta F'$$

$$-\frac{\eta}{2}F' + \frac{1}{\eta}F' = F''$$

$$\left(\frac{1}{\eta} - \frac{\eta}{2}\right)F' = \frac{d}{d\eta}F'$$

$$\left(\frac{1}{\eta} - \frac{\eta}{2}\right)d\eta = \frac{dF'}{F'}$$

$$\ln\eta - \frac{\eta^2}{4} + C = \ln F'$$

$$\exp\left(\ln\eta - \frac{\eta^2}{4} + C\right) = F' = C\,\eta e^{-\eta^2/4}$$

$$C\int \eta \exp\left(-\frac{\eta^2}{4}\right)d\eta + D = F(\eta)$$

$$x^2 = \frac{\eta^2}{4}$$
$$2x\,dx = \frac{\eta}{2}d\eta$$
$$4x\,dx = \eta\,d\eta$$

$$\int \eta\exp\left(-\frac{\eta^2}{4}\right)d\eta = 4\int x\exp(-x^2)\,dx = 4\left(-\frac{1}{2}e^{-x^2}\right) = -2e^{-\eta^2/4}$$

$F(\infty) = 1 \quad -2Ce^{-\eta^2/4} + D = F(\eta)$

$F(0) = 0 \quad -2C + D = 0 \quad C = \frac{D}{2}, D = 1$

$F = 1 - e^{-\eta^2/4}$

$u_\theta = \frac{\Gamma}{2\pi r}\left(1 - e^{-\eta^2/4}\right)$ = Gaussian vortex $\sigma^2 = 4\nu t$

$r \ll \delta$ rigid body rotation

$r \gg \delta$ ideal vortex

Decay line vortex

Gaussian vortex: $u_\theta = \frac{\Gamma}{2\pi r}\left(1 - e^{-r^2/\sigma^2}\right)$

Exercise 3.28. Starting from (3.29), show that the maximum u_θ in a Gaussian vortex occurs when $1+2(r^2/\sigma^2)=\exp(r^2/\sigma^2)$. Verify that this implies $r \approx 1.12091\sigma$.

Solution 3.28. Differentiate the u_θ equation from (3.29) with respect to r and set this derivative equal to zero.

$$\frac{d}{dr}\left(u_\theta(r)\right)=\frac{\Gamma}{2\pi}\frac{d}{dr}\left(\frac{1-\exp\left(-r^2/\sigma^2\right)}{r}\right)=\frac{\Gamma}{2\pi}\left(-\frac{1-\exp\left(-r^2/\sigma^2\right)}{r^2}-\frac{\exp\left(-r^2/\sigma^2\right)}{r}\left(-\frac{2r}{\sigma^2}\right)\right)=0.$$

Eliminate common factors assuming $r \neq 0$.

$$0=-1+\exp\left(-r^2/\sigma^2\right)+\left(2r^2/\sigma^2\right)\exp\left(-r^2/\sigma^2\right)=-1+\left(1+2r^2/\sigma^2\right)\exp\left(-r^2/\sigma^2\right).$$

This can be rearranged to:

$$\exp\left(r^2/\sigma^2\right)=1+2r^2/\sigma^2,$$

which is the desired result. When $r/\sigma \approx 1.12091$, then

$$\exp\left(r^2/\sigma^2\right)=3.51289 \quad\text{and}\quad 1+2r^2/\sigma^2=3.51288,$$

which is suitable numerical agreement.

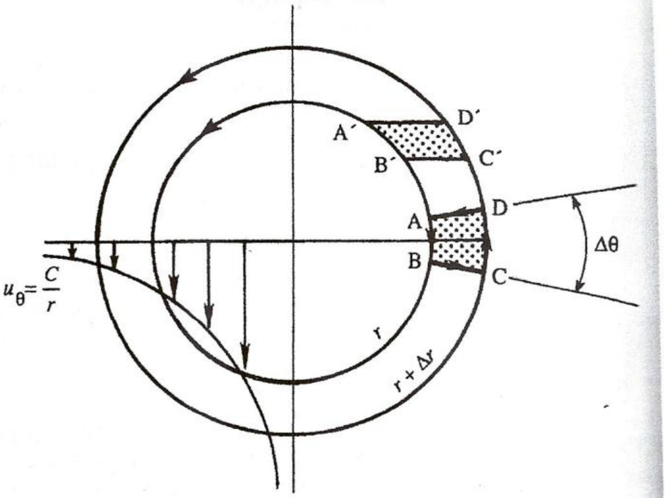


FIGURE 3.17 Irrotational vortex. The streamlines are circular, as for solid body rotation, but the fluid velocity varies with distance from the origin so that fluid elements only deform; they do not spin. The vorticity of fluid elements is zero everywhere, except at the origin where it is infinite.

Instead, real vortices combine elements of the ideal vortex flows described by (3.22) and (3.25). Near the center of rotation, a real vortex's core flow is nearly solid-body rotation, but far from this core, real-vortex-induced flow is nearly irrotational. Two common idealizations of the this behavior are the Rankine vortex defined by:

$$\omega_z(r)=\begin{Bmatrix}\Gamma/\pi\sigma^2=\text{const.} & \text{for } r\le\sigma\\ 0 & \text{for } r>\sigma\end{Bmatrix} \quad\text{and}\quad u_\theta(r)=\begin{Bmatrix}(\Gamma/2\pi\sigma^2)r & \text{for } r\le\sigma\\ \Gamma/2\pi r & \text{for } r>\sigma\end{Bmatrix}, \qquad (3.28)$$

and the Gaussian vortex defined by:

$$\omega_z(r)=\frac{\Gamma}{\pi\sigma^2}\exp(-r^2/\sigma^2), \quad\text{and}\quad u_\theta(r)=\frac{\Gamma}{2\pi r}(1-\exp(-r^2/\sigma^2)) \qquad (3.29)$$

In both cases, σ is a core-size parameter that determines the radial distance where real vortex behavior transitions from solid-body rotation to irrotational-vortex flow. For the Rankine vortex, this transition is abrupt and occurs at $r=\sigma$ where u_θ reaches its maximum. For the Gaussian vortex, this transition is gradual and the maximum value of u_θ is reached at $r/\sigma \approx 1.12091$ (see Exercise 3.28).

Line vortex suddenly introduced into fluid at rest = impulsive rotational start infinitely thin and fast rotating cylinder at r = 0

Exercise 9.34. Suppose a line vortex of circulation Γ is suddenly introduced into a fluid at rest at $t = 0$. Show that the solution is $u_\theta(r,t) = (\Gamma/2\pi r)\exp\{-r^2/4\nu t\}$. Sketch the velocity distribution at different times. Calculate and plot the vorticity, and observe how it diffuses outward.

Solution 9.34. The solution to this problem is very similar to the decay of a line vortex (see Example 9.8). In two-dimensional (r,θ)-polar coordinates, the governing equation is:

$$\frac{\partial u_\theta}{\partial t} = \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right)\right].$$

The boundary conditions on the velocity $u_\theta(r, t)$ are

$$u_\theta(r,0^+) = 0,\ u_\theta(r,\infty) = \Gamma/2\pi r, \text{ and } u_\theta(\infty, t) = 0.$$

In this case the second boundary condition suggests a similarity solution of the form:

$$u_\theta = \frac{\Gamma}{2\pi r} f(\eta) = \frac{\Gamma}{2\pi r} f\left(\frac{r}{\sqrt{\nu t}}\right).$$

For this solution form the time and radial derivatives are:

$$\frac{\partial u_\theta}{\partial t} = \frac{\Gamma}{2\pi r}\frac{df}{d\eta}\frac{\partial\eta}{\partial t} = \frac{\Gamma}{2\pi r}\frac{df}{d\eta}\left(-\frac{\eta}{2t}\right) = -\frac{\Gamma}{2\pi r}\left(\frac{\eta}{2t}\right)\frac{df}{d\eta},\quad \frac{\partial(r u_\theta)}{\partial r} = \frac{\Gamma}{2\pi}\frac{df}{d\eta}\frac{\partial\eta}{\partial r} = \frac{\Gamma}{2\pi}\frac{df}{d\eta}\left(\frac{1}{\sqrt{\nu t}}\right), \text{ and}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right) = \frac{\partial}{\partial r}\left(\frac{\Gamma}{2\pi r}\frac{1}{\sqrt{\nu t}}\frac{df}{d\eta}\right) = -\frac{\Gamma}{2\pi r^2}\frac{1}{\sqrt{\nu t}}\frac{df}{d\eta} + \frac{\Gamma}{2\pi r}\frac{1}{\nu t}\frac{d^2 f}{d\eta^2}.$$

Reassemble the governing equation and divide out the common factor of $\Gamma/2\pi r$:

$$-\left(\frac{\eta}{2t}\right)\frac{df}{d\eta} = -\frac{\nu}{r\sqrt{\nu t}}\frac{df}{d\eta} + \frac{\nu}{\nu t}\frac{d^2 f}{d\eta^2} = -\frac{1}{t}\left(\frac{1}{\eta}\frac{df}{d\eta} - \frac{d^2 f}{d\eta^2}\right).$$

Multiply by t and put the second derivative on the left: $\displaystyle\frac{d^2 f}{d\eta^2} = \left(-\frac{\eta}{2} + \frac{1}{\eta}\right)\frac{df}{d\eta}$.

Integrate to find: $\displaystyle\ln\frac{df}{d\eta} = -\frac{\eta^2}{4} + \ln\eta + const.$ Exponentiate $\displaystyle\frac{df}{d\eta} = e^{const.}\eta\exp\{-\eta^2/4\}$

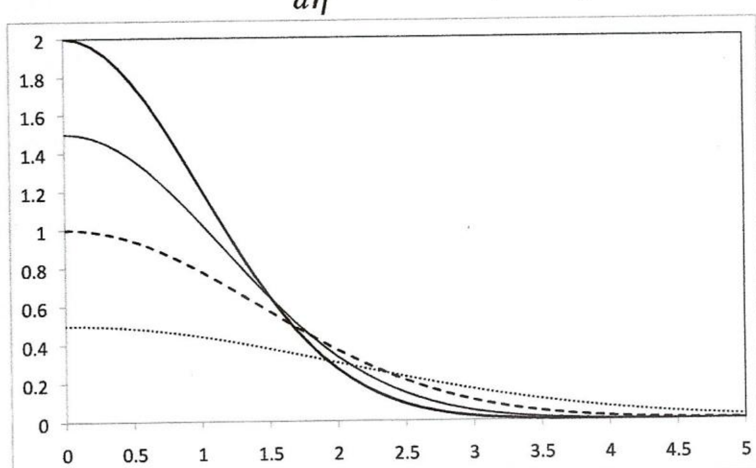
and integrate again:

$$f = A + B\exp\{-\eta^2/4\}.$$

The constants A and B can be determined from the boundary conditions: $f(0) = 1$, and $f(\infty) = 1$; $A = 0$, and $B = 1$. Thus, the velocity field is:

$$u_\theta(r,t) = \frac{\Gamma}{2\pi r}\exp\left\{-\frac{r^2}{4\nu t}\right\}.$$

In this flow the z-component of the vorticity is the only non-zero component.



$$-\ \omega_z(r,t) = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial\theta} = \frac{\Gamma}{4\pi\nu t}\exp\left\{-\frac{r^2}{4\nu t}\right\} = \frac{\Gamma}{4\pi\nu t}\exp\left\{-\frac{\eta^2}{4}\right\}.$$

The plot above shows ω_z (vertical axis) vs. r (horizontal axis) at four different times. With increasing time, the vorticity ar $r = 0$ decreases but it spreads outward in the radial direction.

Stokes 1st: impulsive plate motion

$u = U\left(1 - \text{erf}(\eta)\right) \qquad \eta = \frac{y}{2\sqrt{\nu t}} \qquad \eta_y = \frac{1}{2\sqrt{\nu t}}$

$\omega_z = -u_y = \frac{U}{\sqrt{\pi\nu t}} e^{-\eta^2} = \omega_z(y,t) \;\; \frac{d}{dy}\left(\text{erf}(\eta)\right) = \frac{2}{\sqrt{\pi}} e^{-y^2}$

$\omega_z = f(y,t)$: $\qquad \omega_z(0,0) = \infty \qquad \omega_z(y,0) = 0$

$\frac{u}{U} = 0.05 \Rightarrow \eta = 1.38 \quad \omega_z(0, t > 0) = \frac{U}{\sqrt{\pi\nu t}}$

$\delta = 2.76\sqrt{\nu t} \qquad \omega_z(y > 0, t > 0) = \frac{U}{\sqrt{\pi\nu t}} e^{-\eta^2}$

Diffusion Vortex Sheet

$u = U\,\text{erf}(\eta) \qquad \eta = \dfrac{y}{2\sqrt{\nu t}}$

Same conclusions $\quad \omega = -u_y = -\frac{U}{\sqrt{\pi\nu t}} e^{-\eta^2}$

ω as Stokes 1st

$\tau_w = \mu u_y|_{y=0} = \dfrac{\mu U}{\sqrt{\pi\nu t}}$

$C_f = \dfrac{\tau_w}{\frac{1}{2}\rho U^2} = \dfrac{2}{\sqrt{\pi}}\sqrt{\dfrac{\nu}{U^2 t}}$

Ut = x in spatially developing BL

such that $\quad \sqrt{\frac{\nu}{U^2 t}} = \left(\frac{\nu}{Ux}\right)^{1/2} = Re_x^{-1/2}$

$u = \pm 0.95U \quad \eta = \pm 1.38 \quad \delta = 5.52\sqrt{\nu t}$

$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{\nu}{U^2 t}} \quad t = \frac{x}{U}$

$= \dfrac{2}{\sqrt{\pi}} Re_x^{-1/2}$

$= 1.13\, Re_x^{-1/2}$

Blasius: $\quad C_f = 0.664\, Re_x^{-1/2}$

Similar Stokes 1st and laminar BL.

1. $u_s = U - u_\infty$
2. y > 0 = temporally developing BL with inviscid uniform stream far from wall

Decay ideal line vortex: Oseen vortex

$$u_\theta = \frac{\Gamma}{2\pi r}\left[1 - e^{-r^2/4\nu t}\right] = \frac{\Gamma}{2\pi\sqrt{\nu t}}\left(\frac{1}{\eta}\right)\left[1 - e^{-\eta^2/4}\right]$$

$$\eta = \frac{r}{\sqrt{\nu t}} \quad \eta_r = 1/\sqrt{\nu t}$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) = \frac{\Gamma}{4\pi\nu t} e^{-\eta^2/4}$$

$$r u_\theta = \frac{\Gamma}{2\pi}\left(1 - e^{-\eta^2/4}\right)$$

$$\frac{\partial}{\partial r}(r u_\theta) = -\frac{\Gamma}{2\pi}\left(e^{-\eta^2/4}\right)\left(-\frac{\eta}{2}\times\frac{1}{\sqrt{\nu t}}\right) = \frac{\Gamma}{2\pi}\frac{\eta}{2\sqrt{\nu t}} e^{-\eta^2/4}$$

$$\omega_z = f(r,t)$$

$\omega_z(0,0) = \frac{1}{0} = \infty$ Same behavior Stokes 1st except $\propto t^{-1}$ vs $t^{-1/2}$

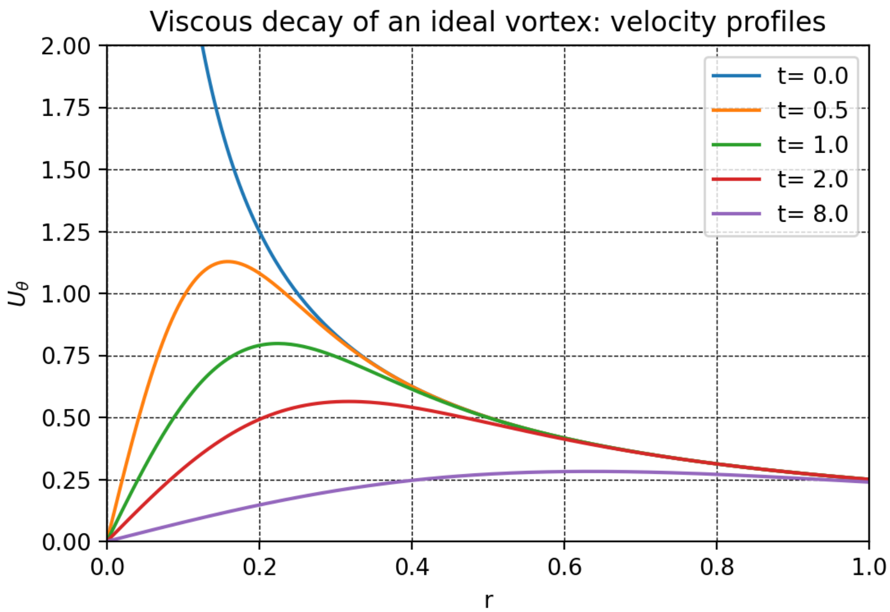
$$\omega_z(r > 0,0) = 0 \qquad \omega_z(0, t > 0) = \frac{\Gamma}{4\pi\nu t}$$

Impulsive line vortex

$$u_\theta = \frac{\Gamma}{2\pi r} e^{-r^2/4\nu t}$$

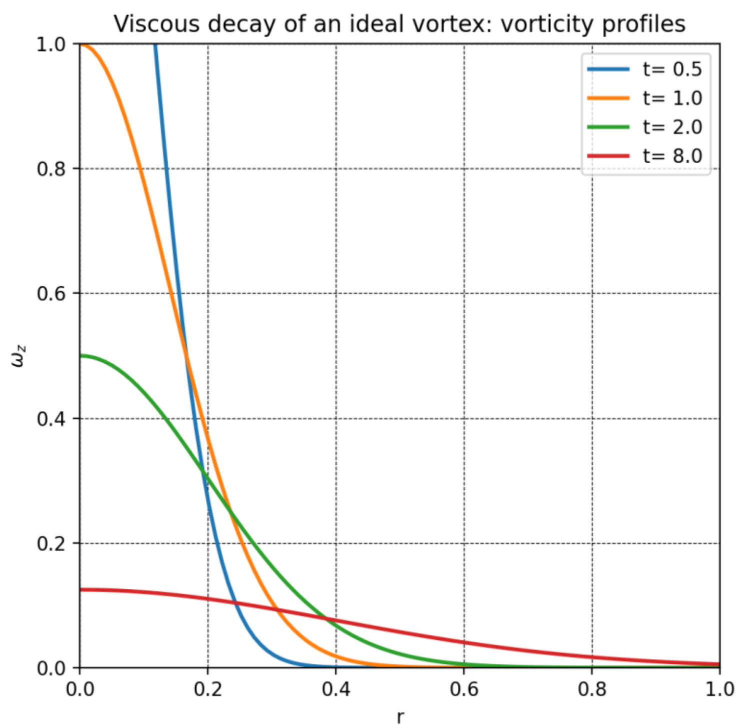
$$\omega_z = -\frac{\Gamma}{4\pi\nu t} e^{-r^2/4\nu t}$$

Same conclusions decay vs impulsive as Stokes 1st vs diffusion vortex sheet



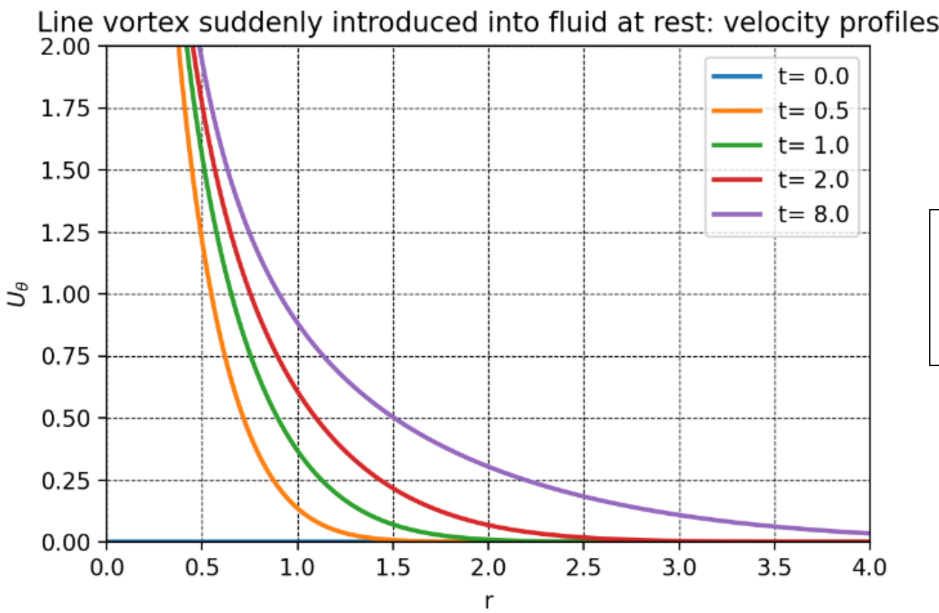
$$u_\theta(0,0) = \infty$$
$$u_\theta(0,t) = 0$$
$$u_\theta(\infty,t) = \frac{\Gamma}{2\pi r} \to 0$$

$$u_\theta = \frac{\Gamma}{2\pi r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right)$$



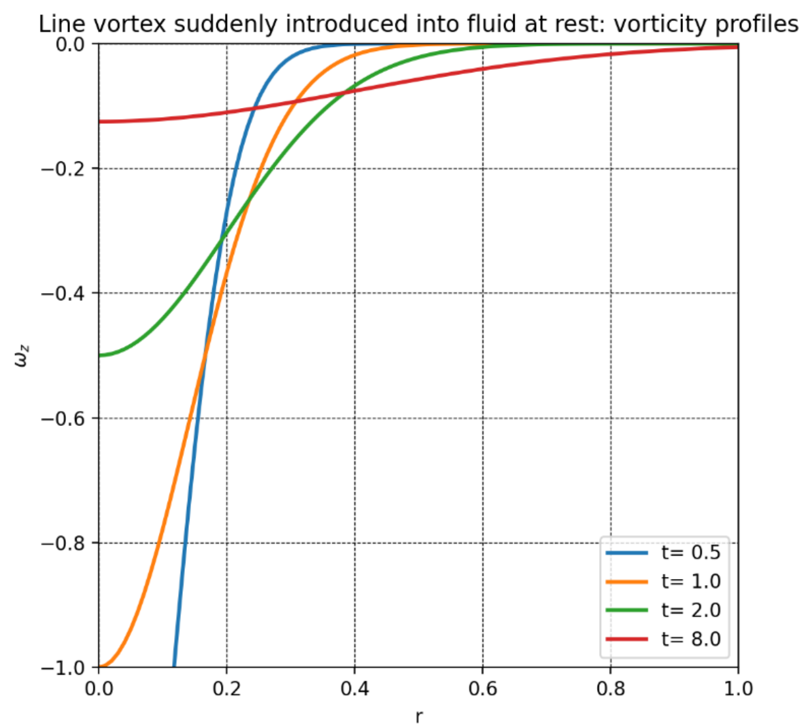
$$\omega_z(0,0) = \infty$$
$$\omega_z(0,t) = \frac{\Gamma}{4\pi\nu t}$$
$$\omega_z(\infty,t) = 0$$

$$\omega_z = \frac{\Gamma}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}}$$



$$u_\theta(0,0) = \infty$$
$$u_\theta(0,t) = \infty$$
$$u_\theta(\infty,t) = 0$$

$$u_\theta = \frac{\Gamma}{2\pi r} e^{-\frac{r^2}{4\nu t}}$$



$$\omega_z(0,0) = -\infty$$
$$\omega_z(0,t) = -\frac{\Gamma}{4\pi\nu t}$$
$$\omega_z(\infty,t) = 0$$

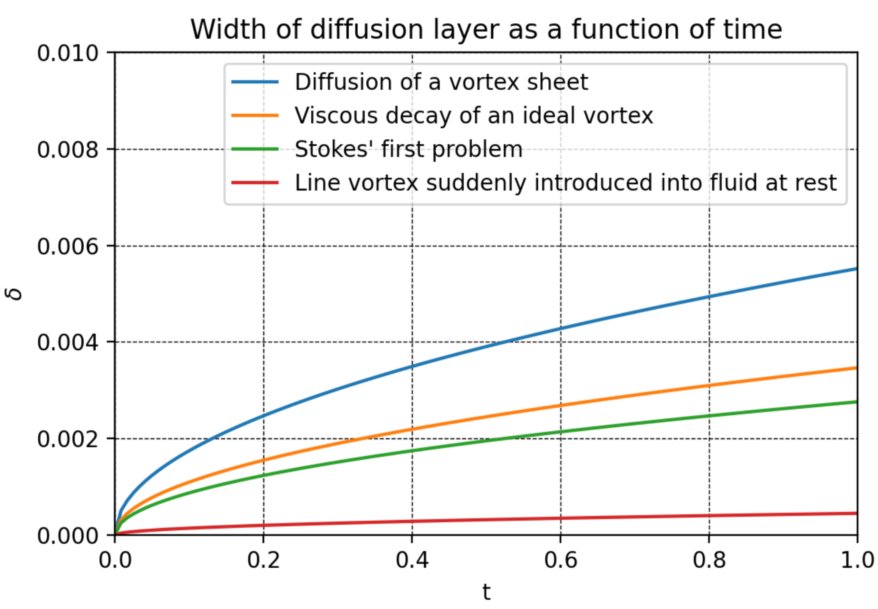
$$\omega_z = -\frac{\Gamma}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}}$$

Width of diffusion layer as a function of time: viscous decay of an ideal vortex

$$u_\theta = \frac{\Gamma}{2\pi r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right)$$
$$\frac{u_\theta}{\frac{\Gamma}{2\pi r}} = 0.95 = 1 - e^{-\frac{r^2}{4\nu t}}$$
$$0.05 = e^{-\frac{r^2}{4\nu t}}$$
$$\log(0.05) \sim -3.0 = -\frac{r^2}{4\nu t}$$
$$r \sim \sqrt{12\nu t} = 3.46\sqrt{\nu t}$$

Width of diffusion layer as a function of time: Line vortex suddenly introduced into fluid at rest

$$u_\theta = \frac{\Gamma}{2\pi r} e^{-\frac{r^2}{4\nu t}}$$
$$\frac{u_\theta}{\frac{\Gamma}{2\pi r}} = 0.95 = e^{-\frac{r^2}{4\nu t}}$$
$$0.95 = e^{-\frac{r^2}{4\nu t}}$$
$$\log(0.95) \sim -0.05 = -\frac{r^2}{4\nu t}$$
$$r \sim \sqrt{0.2\nu t} = 0.45\sqrt{\nu t}$$



	δ	$\dot{\delta}$
Diffusion of a vortex sheet	$5.52\sqrt{\nu t}$	$2.76\sqrt{\nu/t}$
Viscous decay of an ideal vortex	$3.46\sqrt{\nu t}$	$1.73\sqrt{\nu/t}$
Stokes' first problem	$2.76\sqrt{\nu t}$	$1.38\sqrt{\nu/t}$
Sudden line vortex	$0.45\sqrt{\nu t}$	$0.225\sqrt{\nu/t}$

$\dot{\delta} \to 0$ as $t \to \infty$, i.e., rate of diffusion decreases over time

Burgers Vortex

The line vortex viscous spreading can be cancelled by superposing a radial inflow towards the core and a steady flow is obtained.

Consider a $u_\theta(r)$ with axis along z-axis and added to this flow a symmetric radial inflow is added

$$u_r = -ar \qquad \text{a = strength of radial flow}$$

u_r is unbounded at ∞; hence correct solution is local flow near small r

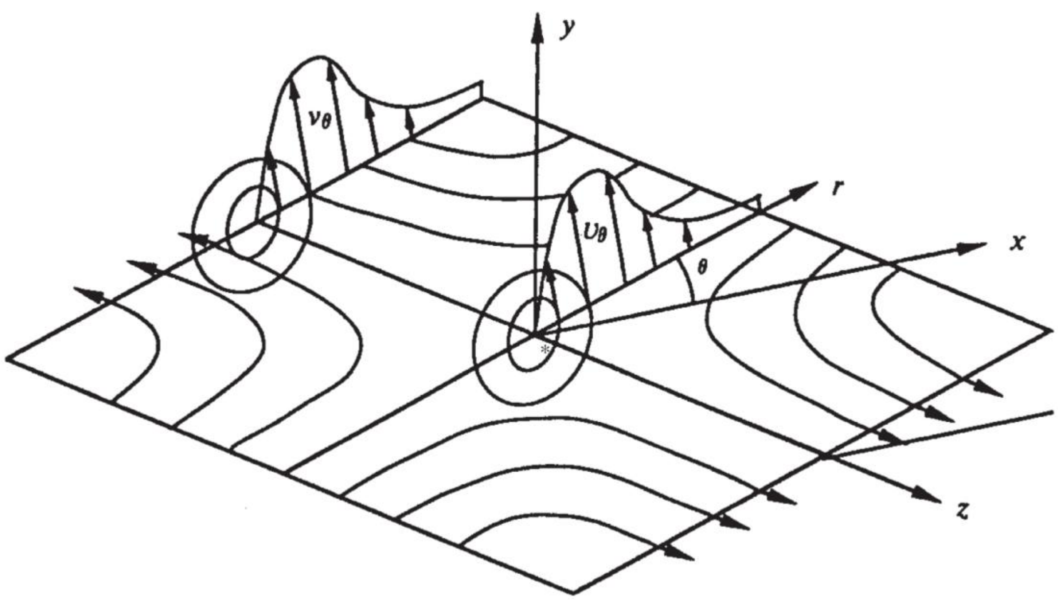


Figure 11.11 Burgers vortex.

Continuity $\frac{\partial u_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = 2a$

$$u_z = 2az$$

$(u_r, u_z) =$ axisymmetric inviscid flow toward stagnation a point at the origin.

Strain rates $\varepsilon_{rr} = \varepsilon_{\theta\theta} = -a$ and $\varepsilon_{zz} = 2a$, i.e., vortex is stretched along its axis at rate a

Assume $u_\theta = u_\theta(r)$ only

$$\nu \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r u_\theta)\right] = -ar\frac{du_\theta}{dr} \qquad (1)$$

$$u_\theta(0) = 0 \qquad u_\theta(\infty) = \frac{\Gamma}{2\pi r}$$

change variable using reduced circulation

$$\gamma = f = \frac{2\pi r u_\theta}{\Gamma}$$

(1) becomes $-a\frac{df}{dr} = \nu\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right)$

next, transform using similarity variable $\eta = \frac{r}{\sqrt{\nu/2a}}$

$$f'' + \left(\frac{1}{2}\eta - \frac{1}{\eta}\right)f' = 0$$

$$f = 1 - \exp\left(-\frac{\eta^2}{4}\right)$$

$$u_\theta = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{2\nu/a}\right)\right]$$

vortex core diffusion is cancelled by radial inflow such that flow is steady state.

Stokes 2nd Problem: Oscillating Plate

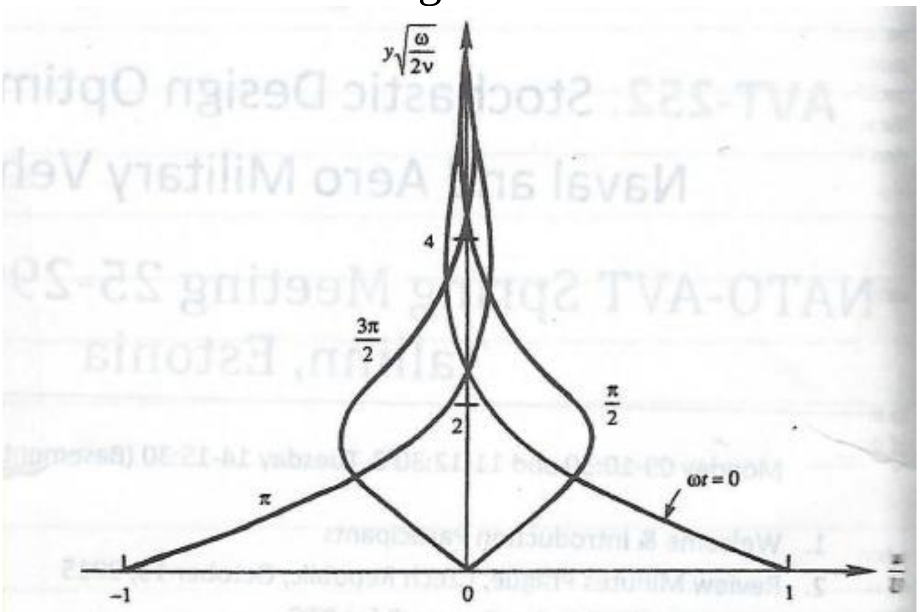


Figure 9.13 Velocity distribution in laminar flow near an oscillating plate. The distributions at $\omega t = 0$, $\pi/2$, π, and $3\pi/2$ are shown. The diffusive distance is of order $\delta = 4\sqrt{\nu/\omega}$.

$$u_t = \nu u_{yy}$$
$$u(0,t) = U\cos\omega t$$
$$u(\infty,t) = 0$$

Steady state periodic solution

$u = e^{i\omega t} f(y)$ f(y) complex

u = real RHS which has phase difference u(0,t)

$i\omega f = \nu f_{yy}$

Equation with constant coefficients has exponential solution of form

$f = e^{ky}$ $k = \sqrt{\frac{i\omega}{\nu}} = \pm(1+i)\sqrt{\frac{\omega}{2\nu}}$ $i = \sqrt{-1}$

$$\sqrt{i} = \pm(1+i)/\sqrt{2}$$

$\therefore\ f = Ae^{-(1+i)y\sqrt{\omega/2\nu}} + Be^{(1+i)y\sqrt{\omega/2\nu}}$

$u = Ae^{i\omega t}e^{-(1+i)y\sqrt{\omega/2\nu}}$ $u(\infty,t) = 0$ $\Rightarrow$ B = 0

$u(0,t) = U\cos\omega t \Rightarrow A = U$

real part: $u = Ue^{-y\sqrt{\omega/2\nu}}\cos\left(\omega t - \underbrace{y\sqrt{\omega/2\nu}}_{phase\ lag\ \ u\ vs\ plate}\right)$

Damped oscillatory motion in y direction

at $y = 4\sqrt{\nu/\omega}$

$u = U\exp\left(-4/\sqrt{2}\right) = 0.06\text{U}$ amplitude

i.e., wall influence confined to $\underbrace{\delta = 4\sqrt{\nu/\omega} \neq f(t)}_{\text{diffusion distance}}$ which is ∝ $\sqrt{\nu}$ and decreases with ω.

Note: $u/U = f(y,t,\omega,\nu)$

Dimensional analysis $u/U = F\left(\omega t,\ y\sqrt{\omega/\nu}\right)$

3 π's and similarity not possible, i.e., cannot be represented using single curve in terms of nondimensional variables. y separated t, whereas for similarity y and t must be combined along with in this case diffusion distance $\delta \neq f(t)$ vs Stokes 1st for which δ increases $\sqrt{t}$. Previous solutions similarity possible since no imposed/specified length or time scales. Stokes 2nd and unsteady fully developed pipe flow have imposed time scale 1/ ω.

Stokes 1st $u_{yy} > 0$ all $y \Rightarrow u_t > 0$ and momentum constantly diffused outward such that δ increases $\sqrt{t}$.

Stokes 2nd u_{yy} and u_t oscillatory in sign and momentum confined such that constant δ.

phase lag relative y = 0

$$\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}\, y\right) = cos\sqrt{\frac{\omega}{2\nu}}\left(\sqrt{2\nu\omega}\, t - y\right) = cosk(ct - y)$$ traveling wave

$k = \frac{2\pi}{\lambda}$ = wave number $\lambda = \frac{2\pi}{k}$ = wavelength $= 2\pi\sqrt{\frac{2\nu}{\omega}}$

$c = \sqrt{2\nu\omega}$ = wave speed $\omega = 1/s$ $\nu = m^2/s$ $\nu/\omega = m^2$

$\exp\left(-\sqrt{\frac{\omega}{2\nu}}\, y\right)$ exponentially damped

Damped viscous wave traveling away from wall. Effect wall motion delayed as per phase difference τ_w vs u_{max}.

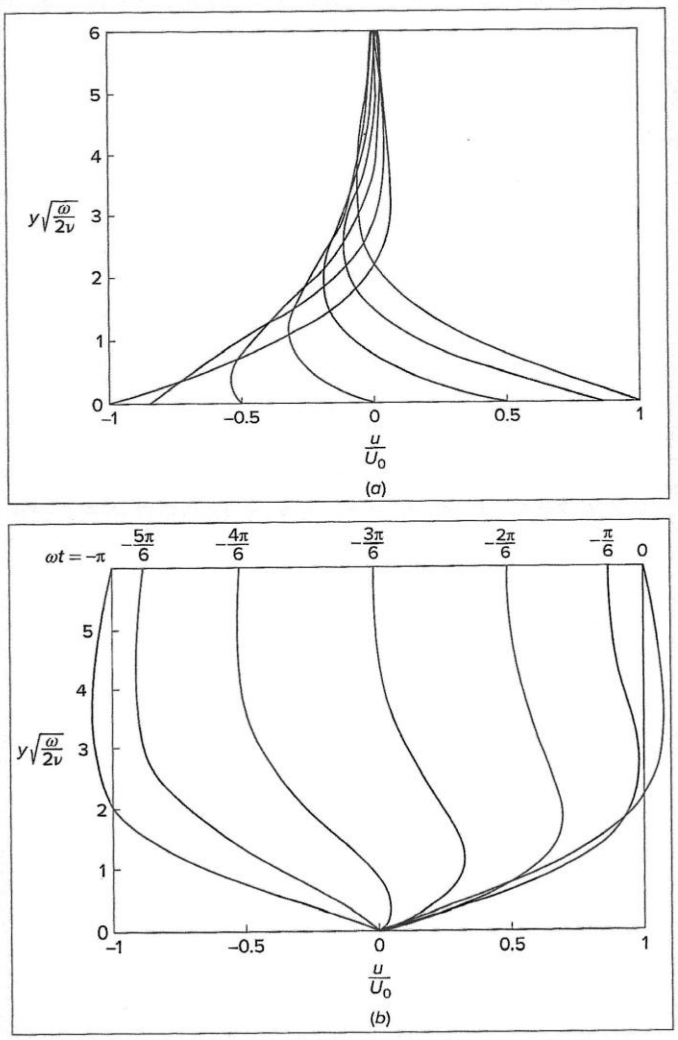


FIGURE 3-22
Stokes' second problem: (*a*) flow above an oscillating plate [Eq. (3-148)]; (*b*) an oscillating fluid above a fixed plate [Eq. (3-149)]. Velocity profiles are shown at different times that are separated by 30° increments over a half period.

Appendix A

Stokes 1st $u = U e^{-\eta}\cos(\omega t - \eta)$ $\eta = y\sqrt{\frac{\omega}{2\nu}}$

$u_{max} = \pm U$ for $\eta = 0$

$\omega t = 0, -\pi$ $\eta_y = \sqrt{\frac{\omega}{2\nu}}$

$\tau_w = \mu u_y\big|_{y=0}$ $u_y = U\left[-e^{-\eta}\eta_y\cos(\omega t-\eta) + e^{-\eta}\sin(\omega t-\eta)\,\eta_y\right]$

$= U e^{-\eta}\eta_y\left[-\cos(\omega t-\eta) + \sin(\omega t-\eta)\right]$

$u_y\big|_{y=0} = U\sqrt{\frac{\omega}{2\nu}}\,\underbrace{\left[-\cos\omega t + \sin\omega t\right]}_{\sqrt{2}\sin(\omega t - \pi/4)}$

$= U\sqrt{\frac{\omega}{\nu}}\sin(\omega t - \pi/4)$ $\cos(\omega t - \frac{3\pi}{4})$

$\tau_w = \mu U\sqrt{\frac{\omega}{\nu}}\cos(\omega t - \frac{3\pi}{4})$ $\frac{Ns}{m^2}\frac{m}{s}\sqrt{\frac{s}{s\,m^2}}$

$\frac{Ns}{m^2}\frac{m}{s}\frac{1}{m} = \frac{N}{m^2}$

lags u_{max} by 135 deg

$u/U = .01$ $e^{-\eta} = .01$ $\eta \sim 4.6 = \delta\sqrt{\frac{\omega}{2\nu}}$

$\delta = 4.6\sqrt{2}\sqrt{\frac{\nu}{\omega}} = 6.5\sqrt{\nu/\omega}$

Shows dependence $\sqrt{\nu}$ & reduces as $\omega\uparrow$

air 20°C $\omega = 1$ Hz $\omega = 2\pi$ rad/s $\delta = 1$ cm

Stokes 2nd problem can alternatively be transformed into flow over fixed wall due to unsteady outer flow caused by oscillating pressure gradient

$p_x = \rho U_0 \omega \sin \omega t$

$\rho u_t = -p_x$ Euler equation

$u_t = -\frac{p_x}{\rho} = -U_0 \omega \sin \omega t$

$u = U_0 \cos \omega t$

$u = U_0 \sin(\omega t + \pi/2)$ leads p_x by 90°

Outer inviscid flow

$\rho u_t = -p_x + \mu u_{yy}$

$u(0, t) = 0$

$u(\infty, t) = U_0 \cos \omega t$

Inner viscous flow, which is referred to as Stokes layer

Solution same as negative previous solution plus outer flow

$$u = \underbrace{U_0 \cos \omega t}_{\text{inviscid part}} \underbrace{- U_0 \exp\left(-\sqrt{\frac{\omega}{2\nu}}\, y\right) \cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}\, y\right)}_{\text{viscous part}}$$

Can also be written as single wave with different amplitude and phase, as per Appendix B.

relative $u = U_0 \cos \omega t$

near $\sqrt{\frac{\omega}{2\nu}}\, y = 2.36\ (0.75\pi)$ which is near 0.38λ

Solution shows *phase leads* and $\overbrace{\text{overshoot}}$ at $0, -\pi$

low momentum fluid near wall responds first to p_x

At $y = 0$ μu_{yy} counteracts p_x so that $u = 0$ satisfying no slip condition.

Away from wall μu_{yy} decays exponentially.

$\begin{cases}\text{Intermediate region } p_x \text{ and } \mu u_{yy} \text{ in} \\ \text{phase such that overshoot occurs}\end{cases}$

p_x transmitted instantaneously vs
μu_{yy} transmitted by viscous diffusion

→ called Richardson overshoot in pipe flow with oscillating p_x

Stern, F., Choi, J.E., and Hwang, W.S., "Effects of Waves on the Wake of a Surface-Piercing Flat Plate: Experiment and Theory," Journal of Ship Research, Vol. 37, No. 2, June 1993, pp. 102 – 118.

Choi, J.-E., Sreedhar, M., and Stern, F., "Stokes Layers in Horizontal-Wave Outer Flows," ASME J. Fluids Eng., Vol. 118, September 1996, pp. 537 – 545.

Paterson, E.G. and Stern, F., "Computation of Unsteady Viscous Marine-Propulsor Blade Flows - Part 1: Validation and Analysis," ASME J. Fluids Eng., Vol. 119, March 1997, pp. 145 – 154.

Paterson, E.G. and Stern, F., "Computation of Unsteady Viscous Marine-Propulsor Blade Flows - Part 2: Parametric Study," ASME J. Fluids Eng, Vol. 121, March 1999, pp. 139 – 147.

Additional discussion Stokes 2nd problem

(1) moving plate

$$u = Ue^{-y\sqrt{\omega/2\nu}} \underbrace{\cos\left(\omega t - y\sqrt{\omega/2\nu}\right)}_{\cos[k(ct-y)]}$$

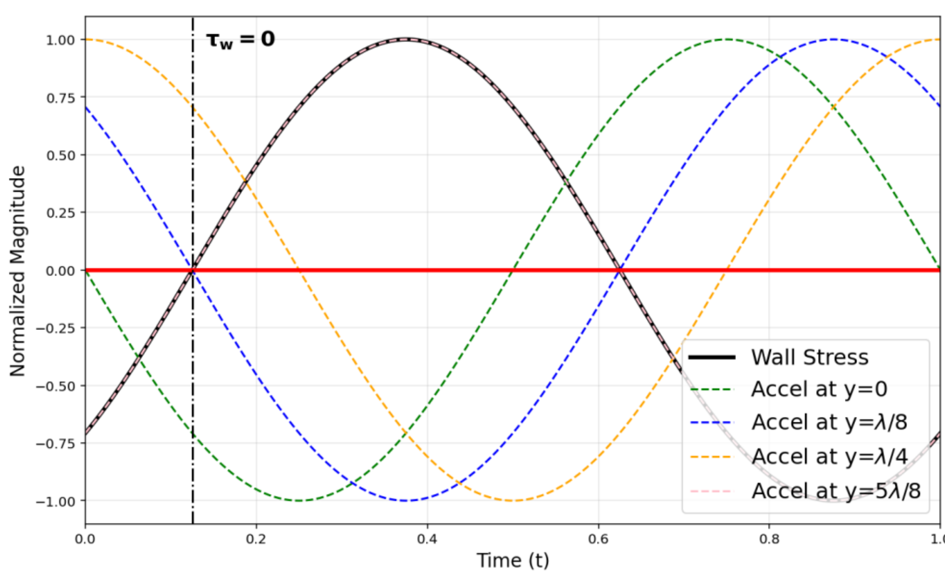
$$k = \frac{2\pi}{\lambda} \qquad \lambda = 2\pi\sqrt{\frac{2\nu}{\omega}} \quad c = \sqrt{2\nu\omega}$$

$\delta = 4\sqrt{\nu/\omega}$ vs. Stokes 1st problem where viscous diffusion grows in time $\delta = 2.76\sqrt{\nu t}$.

Damped viscous wave moving away from the wall. Note that the lag of the acceleration vs. the wall shear stress depends on the distance from the wall.

$$\frac{\partial u}{\partial t} = -\omega U_0 e^{-y\sqrt{\omega/2\nu}} \sin\left(\omega t - y\sqrt{\omega/2\nu}\right) = \nu u_{yy}$$

$$\tau_w = \mu U cos\left(\omega t - \frac{3\pi}{4}\right)$$



(2) oscillating outer flow

$$u = U_0 \cos \omega t - U_0 e^{-y\sqrt{\omega/2\nu}} \cos\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

note overshoot at $y\sqrt{\omega/2\nu} = 2.36\ (0.75\pi)$, i.e., near 0.38λ

$$\frac{\partial u}{\partial t} = -\omega U_0 \sin \omega t + \omega U_0 e^{-y\sqrt{\omega/2\nu}} \sin\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

$$= -\frac{p_x}{\rho} + \nu u_{yy}$$

$$-\frac{p_x}{\rho} = -U_0 \omega \sin \omega t$$

$$u_y = -U_0 \left(-\sqrt{\frac{\omega}{2\nu}}\right) e^{-y\sqrt{\omega/2\nu}} \cos\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

$$+ U_0 e^{-y\sqrt{\omega/2\nu}} \left(-\sqrt{\frac{\omega}{2\nu}}\right) \sin\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

$$u_{yy} = -U_0 \frac{\omega}{2\nu} e^{-y\sqrt{\omega/2\nu}} \cos\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

$$+ U_0 e^{-y\sqrt{\omega/2\nu}} \frac{\omega}{2\nu} \sin\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

$$+ U_0 e^{-y\sqrt{\omega/2\nu}} \frac{\omega}{2\nu} \sin\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

$$+ U_0 e^{-y\sqrt{\omega/2\nu}} \frac{\omega}{2\nu} \cos\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

$$u_{yy} = 2U_0 \left(\frac{\omega}{2\nu}\right) e^{-y\sqrt{\omega/2\nu}} \sin\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

$$\nu u_{yy} = \omega U_0 e^{-y\sqrt{\omega/2\nu}} \sin\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

Large *y* recovers Euler equation

At y = 0

$$-\omega U_0 \sin \omega t + \omega U_0 \sin \omega t = -\omega U_0 \sin \omega t + \omega U_0 \sin \omega t$$

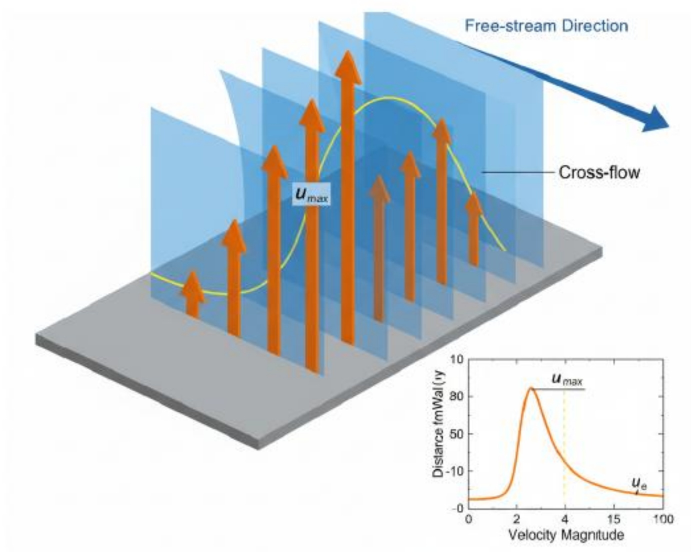
At the wall $u_t = 0$ and pressure gradient counterbalanced by viscous force

Intermediate region pressure and viscous combine to accelerate fluid to higher velocities than that due to pressure forces alone. The net viscous stress created at the wall diffuses and attenuates into the flow and after half cycle later combines/adds with pressure gradient to create overshoot.

Results highlight: p_x transmitted instantaneously, whereas viscous forces are transmitted by viscous diffusion i.e. have diffusive time scale. Viscous forces are not always damping effect:

(1) μ leads to instability flat plate BL

(2) many 3D skewed BL have u_{max} within BL



The figure above illustrates the development of a **3D skewed boundary layer**. In such flows, the velocity vector rotates in direction as distance from the surface increases, often creating a **velocity maximum (u_{max})** within the boundary layer thickness. DSpace@MIT +1

In this 3D state, the interaction between the primary flow and lateral pressure gradients creates a cross-flow component. This cross-flow can cause the local velocity magnitude to overshoot the free-stream velocity (u_e), meaning the fastest fluid is found inside the boundary layer rather than at its outer edge. IntechOpen +2

Stokes' second problem: analysis of the extrema of the velocity field

Velocity field:

$$U = U_0 \cos(\omega t) - U_0 e^{-\sqrt{\frac{\omega}{2\nu}}y} \cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right)$$

Assume $-\sqrt{\frac{\omega}{2\nu}}y = x$

$$U = U_0 \cos(\omega t) - U_0 e^{x} \cos(\omega t + x)$$

$$\frac{dU}{dx} = -U_0 e^{x} \cos(\omega t + x) + U_0 e^{x} \sin(\omega t + x) = 0$$

Divide by $U_0 e^{x}$:

$$-\cos(\omega t + x) + \sin(\omega t + x) = 0$$

$$\tan(\omega t + x) = 1$$

$$\omega t + x = \frac{\pi}{4} + k\pi \rightarrow x = \frac{\pi}{4} + k\pi - \omega t$$

Where $k = 0, \pm 1, \pm 2 \ldots \pm \infty$. Substitute back for $\sqrt{\frac{\omega}{2\nu}}y$:

$$-\sqrt{\frac{\omega}{2\nu}}y = \frac{\pi}{4} + k\pi - \omega t$$

The condition for the location of the extrema is:

$$\sqrt{\frac{\omega}{2\nu}}y = \omega t - \frac{\pi}{4} - k\pi$$

Therefore, the velocity field has multiple local maxima/minima. For example, when $\omega t = 0$, the locations of the local extrema are:

$$\sqrt{\frac{\omega}{2\nu}}y = -\frac{\pi}{4} - k\pi$$

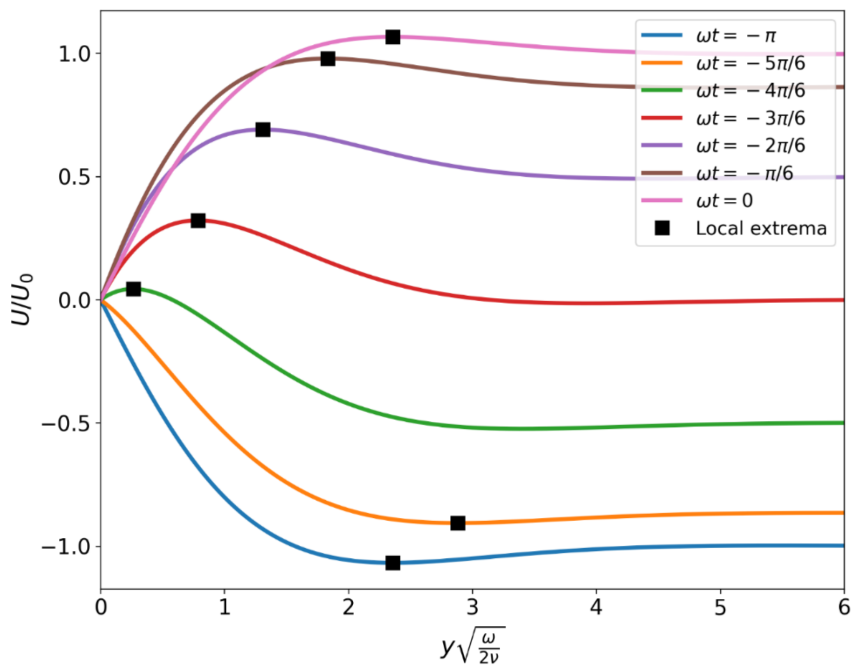
i.e.,

$$\sqrt{\frac{\omega}{2\nu}}y = +\infty, \ldots, \frac{7}{4}\pi, \frac{3}{4}\pi, -\frac{\pi}{4}, -\frac{5}{4}\pi, -\frac{9}{4}\pi, \ldots, -\infty$$

For $k = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots, +\infty$.

The extrema for $\sqrt{\frac{\omega}{2\nu}}y > 2\pi$ are difficult to see due to the damping effect of the exponential function.

If we only consider $0 < \sqrt{\frac{\omega}{2\nu}} y < 6$, the local extrema are shown in the figure below.



ωt	Local extrema		
	$y\sqrt{\frac{\omega}{2\nu}}$	U/U_0	$\lvert U - U_\infty \rvert$
$-\pi$	$\frac{3\pi}{4} = 2.356 = \frac{3}{8}\lambda$	-1.067	0.067
$-5\pi/6$	$\frac{11\pi}{12} = 2.880 = \frac{11}{24}\lambda$	-0.906	0.040
$-4\pi/6$	$\frac{\pi}{12} = 0.262 = \frac{1}{24}\lambda$	0.044	0.544
$-3\pi/6$	$\frac{\pi}{4} = 0.785 = \frac{1}{8}\lambda$	0.322	0.322
$-2\pi/6$	$\frac{5\pi}{12} = 1.309 = \frac{5}{24}\lambda$	0.691	0.191
$-\pi/6$	$\frac{7\pi}{12} = 1.833 = \frac{7}{24}\lambda$	0.979	0.113
0	$\frac{3\pi}{4} = 2.356 = \frac{3}{8}\lambda$	1.067	0.067

For $-\frac{4}{6}\pi < \omega t < 0$, the velocity field shows a local maximum, which moves towards smaller y when ωt increases its absolute value. For $-\pi < \omega t < \frac{4}{6}\pi$, the local maximum moves to negative values of y, i.e., a region which is not physically interesting. Therefore, the next extremum is a minimum, as shown in the figure.

1) The overshoot is located where the pressure gradient and viscous term have the same sign.
2) The y $-$location and amount of the overshoot depends on the value of ωt.
3) The y-location should depend on the travelling wave concept.
4) Explain the physics of the y-location and the amount of the maximum.
5) All of the above need to be compared with Panton's discussion.

Explanation Richardson effect, i.e., velocity overshoot:

At $y = 0$, $u_t = 0$ and $-p_x/\rho$ balances $\nu\, u_{yy}$.

Away from wall oscillatory $\nu\, u_{yy}$ decays $e^{-y\sqrt{\frac{\omega}{2\nu}}}$.

Intermediate region complex, since $-p_x$ instantaneous whereas $\nu\, u_{yy}$ has phase lag $-y\sqrt{\omega/2\nu}$ (at wall phase lag = 0). Close to the wall $-p_x$ and $\nu\, u_{yy}$ are in phase (have same sign), which creates overshoot; however, at $\omega t = 0$ *and* $-\pi$ note that $-p_x = 0$.

Incompressible acoustic air waves near wall with Stokes layer: 200 Hz, $\nu = 0.15$ cm^2/s, $\delta = 4.5(2\nu/\omega)^{1/2} = 1.5$ mm.

Examples of Stokes Layers:

- **Stokes' Second Problem (Oscillating Wall):** A flat plate moving sinusoidally back and forth in a viscous fluid, creating a boundary layer of thickness.
- **Fluid Near an Oscillating Bed:** Fluid motion near the seabed caused by surface water waves, creating a characteristic oscillatory layer.
- **Sound Waves Near Walls:** The oscillating velocity field of a sound wave near a rigid solid surface creates a Stokes boundary layer.
- **Industrial/Biological Flows:** Oscillatory flows in pumps, blood pumping, and mucus flow in respiration.
- **Complex Fluids:** Stokes layers in elasto-viscoplastic, thixotropic, and shear-thickening fluids, which exhibit unique behaviors like jamming during oscillation.
- **Experimental Dusty Plasma:** A two-dimensional, strongly coupled dusty plasma under localized, sinusoidal laser shear.

$U_\infty = U_0 \cos(\omega t)$ represents the velocity in the far-field, i.e., inviscid solution.

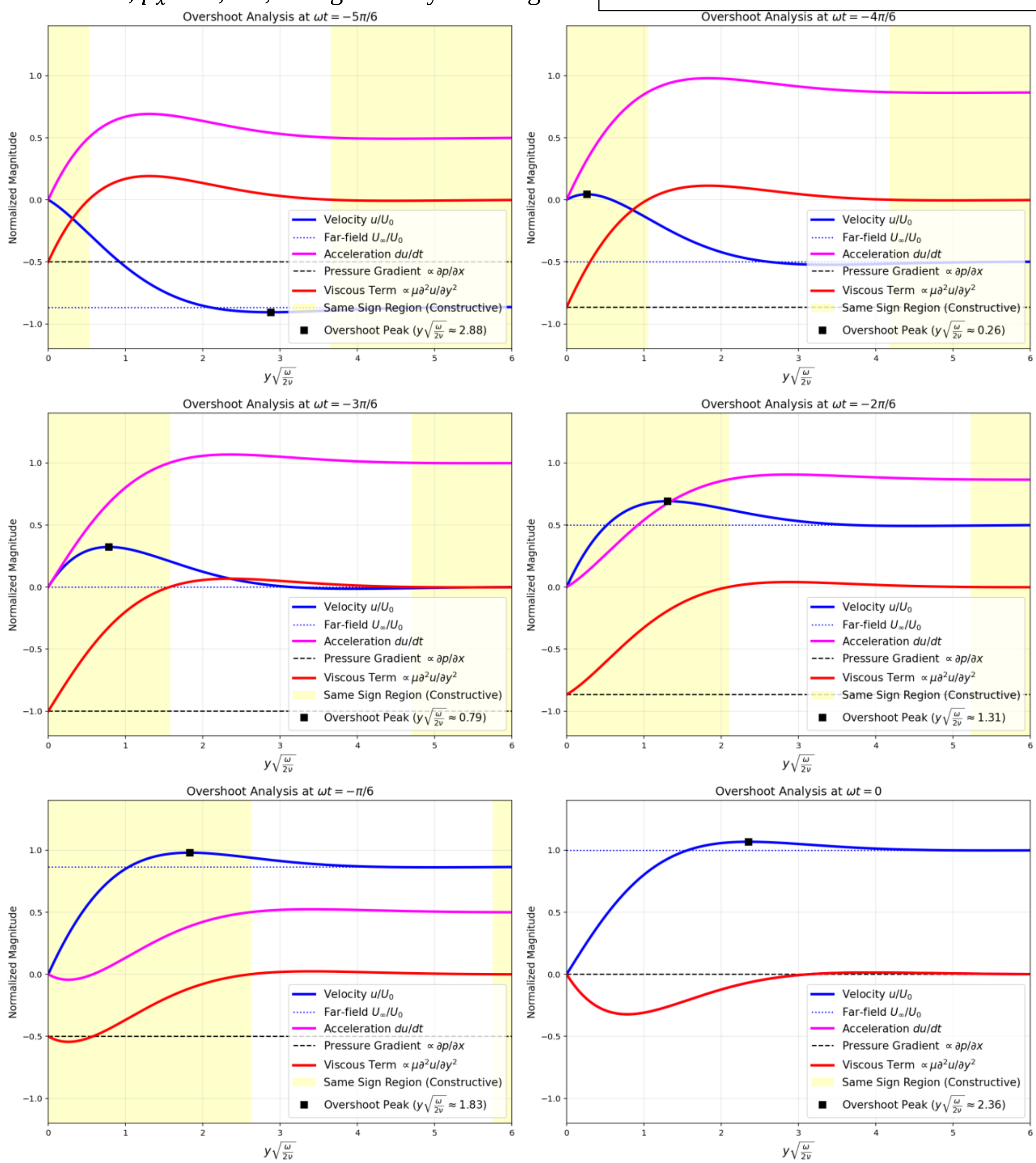
$$u = U_0\left(\cos(\omega t) - e^{-\sqrt{\frac{\omega}{2\nu}}y}\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right)\right)$$

$$\frac{du}{dt} = U_0\left(-\omega\sin(\omega t) + \omega e^{-\sqrt{\frac{\omega}{2\nu}}y}\sin\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right)\right)$$

$$\nu u_{yy} = \omega U_0 e^{-y\sqrt{\omega/2\nu}}\sin\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

In far-field $\nu u_{yy} \to 0$, such that $\frac{du}{dt} - p_x = 0$

For $\omega t = -5\pi/6$, the local extrema do not appear in the yellow region because it's a local minimum. For $\omega t = 0$, $p_x = 0$, i.e., no sign → no yellow region.



Shows that overshoots occur in regions $-p_x$ and νu_{yy} are additive and low momentum.

EXAMPLE 9.10

Show that $\dot{w}$ = the rate of work (per unit area) done on the fluid by the oscillating plate is balanced by $\dot{e}$ = the viscous dissipation of energy (per unit area) in the fluid above the plate.

Solution

The rate of work (per unit area) done on the fluid by the moving plate is the product of the shear stress on the fluid, τ_w, and the plate velocity, $U\cos(\omega t)$:

$$\dot{w} = \tau_w U\cos(\omega t) = -\mu(\partial u/\partial y)_{y=0}\cdot U\cos(\omega t),$$

The negative sign appears because the outward normal from the fluid points downward on the surface of the plate. Differentiating (9.38) with respect to y, leads to:

$$\frac{\partial u}{\partial y} = U\sqrt{\frac{\omega}{2\nu}}\exp\left\{-y\sqrt{\frac{\omega}{2\nu}}\right\}\left[-\cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right) + \sin\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right)\right],$$

and evaluating the result at $y = 0$ produces:

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = U\sqrt{\frac{\omega}{2\nu}}[-\cos(\omega t) + \sin(\omega t)].$$

Thus, the time-average rate of work (per unit area) done by the plate on the fluid is:

$$\overline{\dot{w}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\tau_w U\cos(\omega t)dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} -\mu U\sqrt{\frac{\omega}{2\nu}}[-\cos(\omega t) + \sin(\omega t)]U\cos(\omega t)dt = \mu\frac{U^2}{2}\sqrt{\frac{\omega}{2\nu}}$$

where $2\pi/\omega$ is the period of the plate's oscillations.

From (4.58), the rate of dissipation of fluid kinetic energy per unit volume is $\tau_{ij}S_{ij}$, which reduces to $2\mu S_{ij}S_{ij}$ for an incompressible viscous fluid. Thus, the time-average energy dissipation rate (per unit area) above the plate will be:

$$\overline{\dot{e}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_0^{\infty} 2\mu S_{ij}S_{ij}dydt = \frac{\omega}{\pi}\mu\int_0^{2\pi/\omega}\int_0^{\infty}\left(S_{xy}^2 + S_{yx}^2\right)dydt = \frac{\omega}{2\pi}\mu\int_0^{2\pi/\omega}\int_0^{\infty}\left(\frac{\partial u}{\partial y}\right)^2 dydt,$$

since the only strain-rate component in this flow is $S_{xy} = S_{xy} = (1/2)(\partial u/\partial y)$. The final result is easiest to obtain by performing the time average first:

$$\frac{\omega}{2\pi}\mu\int_0^{2\pi/\omega}\left(\frac{\partial u}{\partial y}\right)^2 dt = \mu U^2\frac{\omega}{2\nu}\exp\left\{-2y\sqrt{\frac{\omega}{2\nu}}\right\}.$$

This leaves the vertical integral:

$$\overline{\dot{e}} = \int_0^{\infty}\mu U^2\frac{\omega}{2\nu}\exp\left\{-y\sqrt{\frac{2\omega}{\nu}}\right\}dy = \mu U^2\frac{\omega}{2\nu}\sqrt{\frac{\nu}{2\omega}} = \mu\frac{U^2}{2}\sqrt{\frac{\omega}{2\nu}},$$

and this matches the time-averaged result for $\dot{w}$. Thus, the average rates of work input and energy dissipation are equal. They are not instantaneously equal, so the fluid's kinetic energy (per unit area) fluctuates, but it does not grow without bound.

Unsteady Fully Developed Pipe Flow

$u = u(r,t) \qquad v = w = 0$

Satisfies continuity and momentum equation reduces to

$$\rho\, u_t = \underbrace{-\hat{p}_x}_{f(t)\text{ only}} + \mu\left(u_{rr} + \frac{1}{r}u_r\right)$$

which is analogous to the linear heat conduction equation with a sources term.

1. Starting flow: fluid accelerates from rest under impulsive action of constant $\hat{p}_x$, which induces axial flow gradually approaching steady Poiseuille profile.

Define $\hat{r} = r/r_o$ where r_o is the pipe radius

Initial condition $\qquad u(\hat{r},0) = 0$

No slip condition $\qquad u(1,t) = 0$

Large t fully developed flow $\quad u(\hat{r},\infty) = u_{\max}(1-\hat{r}^2)$

Taking advantage of linearity, the variables can be separated by subtracting the fully developed flow solution $u' = u - u(\hat{r},\infty)$, which removes inhomogeneity $\hat{p}_x$ and converts solution to form $u = J_0(\lambda\hat{r})\exp\left(-\frac{\lambda^2 \nu t}{r_0^2}\right)$. No slip requires $\lambda = \lambda_n$= zeroes of Bessel function. Since J_0 is not a paraboloid must sum over $J_0(\lambda_n \hat{r})$ and use orthogonality to obtain:

$$\frac{u}{\underbrace{u_{max}}_{=\frac{r_0^2}{4\mu}(-\widehat{p_x})}} = (1-\hat{r}^2) - \sum_{n=1}^{\infty} \frac{8J_0(\lambda_n \hat{r})}{\lambda_n^3 J_1(\lambda_n)} \exp\left(-\frac{\lambda_n^2 \nu t}{r_0^2}\right)$$

TABLE 3-2

First ten roots of the Bessel function J_0†

n	λ_n	$J_1(\lambda_n)$
1	2.4048	0.5191
2	5.5201	−0.3403
3	8.6537	0.2715
4	11.7915	−0.2325
5	14.9309	0.2065
6	18.0711	−0.1877
7	21.2116	0.1733
8	24.3525	−0.1617
9	27.4935	0.1522
10	30.6346	−0.1442

†For $n > 10$: $\lambda_n \approx \frac{(4n-1)\pi}{4}$ $\quad J_1(\lambda_n) \approx (-1)^{n+1}\left(\frac{2}{\pi\lambda_n}\right)^{1/2}$

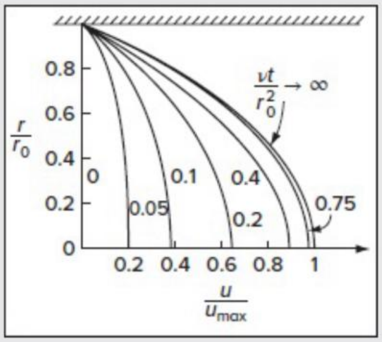


FIGURE 3-16

Instantaneous velocity profiles for starting flow in a pipe, Eq. (3.96). [*After Szymanski* (*1932*).]

https://en.wikipedia.org/wiki/Bessel_function

(1) initially BL effect occurs near wall and central core potential flow accelerates uniformly

(2) for $\underbrace{t^* = .75}$ flow is essentially parabolic $\left(t^* = \frac{\nu t}{r_0^2}\right)$, which can be used to estimate time for flow to respond to sudden change in $f(\nu, {r_0}^2)$, small r_0 large ν develop rapidly

$D = 1$ cm

$\nu_{air} = 1.5 \times 10^{-5}$ m^2/s

$t^* = .75$

$t = 1.25$ s

vs SAE oil $t = .06$ s

∴ for laminar small r_0 pipe flow with $\hat{p}_x = f(t)$ may use quasi-steady assumption.

2. Oscillatory pipe flow

$$-\frac{1}{\rho}\,\hat{p}_x = Ke^{i\omega t} \qquad e^{i\omega t} = \cos\,\omega t + i\sin\,\omega t \qquad i = \sqrt{-1}$$

No slip and steady oscillation condition, i.e., neglect transient and start up, leads to Bessel function with imaginary argument.

$$u(r,t) = \frac{K}{i\omega}\left[1 - \frac{J_0\left(ir\sqrt{\frac{\mathrm{i}\omega}{\nu}}\right)}{J_0\left(ir_0\sqrt{\frac{\mathrm{i}\omega}{\nu}}\right)}\right]e^{i\omega t} \qquad ii^{1/2} = i^{(3/2)} \quad \underbrace{\frac{K}{i\omega} = -\frac{iK}{\omega}}_{as\ per\ Kundu}$$

Where identity $J_0(iz)$=$J_0(z)$ used. Can be solved numerically; however, two overlapping approximate solutions useful to obtain general behavior:

$$J_0(z) \approx 1 - \frac{z^2}{4} + \frac{z^4}{64} + \cdots \quad (z \ll 1)$$

$$J_0(z) \approx \frac{\sqrt{2}}{\pi z}\cos\left(z - \frac{\pi}{4}\right) \quad (z \gg 1)$$

Proper nondimensional variables are:

$$r = \frac{\mathrm{r}}{\mathrm{r}_0}, \quad \underbrace{\omega' = \frac{\omega r_0^2}{\nu}}_{\mathrm{Re}_k = \text{kinetic Re}} \qquad u' = \frac{u}{u_{max}}, \quad u_{max} = \frac{Kr_0^2}{4\nu}$$

u_{max} centerline u for steady Poiseuille flow $\widehat{P_x} = -\rho k$

= measure viscous effects in oscillatory flows

$\omega' > 2000$ transition turbulence

related $\lambda_s = \frac{1}{2}\sqrt{\omega'}$ Stokes # and $\alpha = \sqrt{\omega'}$ Womersley #

using $J_0(z)$ expansions:

$\omega' \ll 1$

$$\frac{u(r,t)}{u_{max}} = (1 - r'^2)\cos\omega t + \frac{\omega'}{16}(r'^4 + 4r'^2 - 5)\sin\omega t + O(\omega'^2)$$

$\omega' \gg 1$

$$\frac{u(r,t)}{u_{max}} = \frac{4}{\omega'}\left[\sin\omega t - \frac{e^{-B}}{r'}\sin(\omega t - B)\right] + O(\omega'^{-2})$$

$B = (1 - r')\sqrt{\omega'/2}$

$\omega' \ll 1$

1st term: since $\hat{p}_x \propto \cos\ \omega t$, u ~ quasi-steady, i.e. Poiseuille flow, in phase with slowly varying $\hat{p}_x$. 2nd term: lag which reduces and centerline velocity.

$\omega' \gg 1$

u lags $\hat{p}_x$ by $\pi/2$ and u centerline < u_{max}. However, near wall region high u which can be seen by averaging solution over one cycle:

mean square $\bar{u^2}/(K^2/(2\omega^2)) = 1 - \frac{2}{\sqrt{r'}}e^{-B} + \frac{e^{-2B}}{r'}$

Overshoot occurs when $\cos\ B + \sin\ B \approx e^{-B}$, i.e., $B = 2.2841$

and $r' = 1 - \frac{3.23}{\sqrt{\omega'}}$.

Speed overshoot ratio:

$$\frac{\overline{u^2}}{(K^2/2\omega^2)} = \frac{\overline{u^2}}{\left(8u_{max}^2/\omega'^2\right)} = 1.143685 + \frac{0.248626}{\sqrt{\omega'}} + O((\omega')^{-1})$$

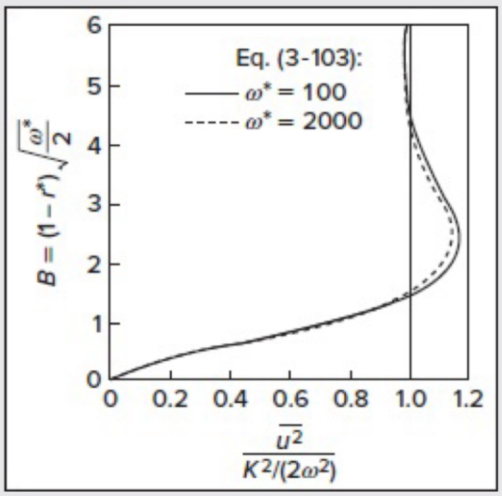


FIGURE 3-17

The near-wall velocity overshoot (Richardson's annular effect) due to an oscillatory pressure gradient.

Overshoot reduces slightly as $Re_K = \omega'$ increase from 100 to 2000 Overshoot = Richardson annular effect. Verified both experimentally and theoretically. Appendix C.

The **Stokes number** (**Stk**), named after George Gabriel Stokes, is a dimensionless number characterising the behavior of particles suspended in a fluid flow. The Stokes number is defined as the ratio of the characteristic time of a particle (or droplet) to a characteristic time of the flow or of an obstacle, or

$$\mathrm{Stk} = \frac{t_0\, u_0}{l_0}$$

where t_0 is the relaxation time of the particle (the time constant in the exponential decay of the particle velocity due to drag), u_0 is the fluid velocity of the flow well away from the obstacle, and l_0 is the characteristic dimension of the obstacle (typically its diameter) or a characteristic length scale in the flow (like boundary layer thickness).[1] A particle with a low Stokes number follows fluid streamlines (perfect advection), while a particle with a large Stokes number is dominated by its inertia and continues along its initial trajectory.

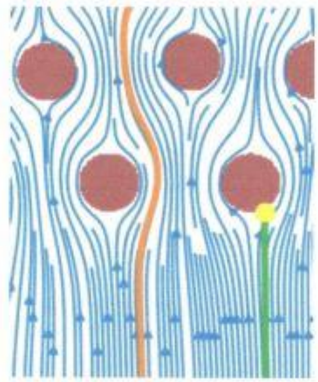


Illustration of the effect of varying the Stokes number. Orange and green trajectories are for small and large Stokes numbers, respectively. Orange curve is trajectory of particle with Stokes number less than one that follows the streamlines (blue), while green curve is for a Stokes number greater than one, and so the particle does not follow the streamlines. That particle collides with one of the obstacles (brown circles) at point shown in yellow.

In the case of Stokes flow, which is when the particle (or droplet) Reynolds number is less than unity, the particle drag coefficient is inversely proportional to the Reynolds number itself. In that case, the characteristic time of the particle can be written as

$$t_0 = \frac{\rho_p d_p^2}{18\mu_g}$$

where ρ_p is the particle density, d_p is the particle diameter and μ_g is the fluid dynamic viscosity.[2]

In experimental fluid dynamics, the Stokes number is a measure of flow tracer fidelity in particle image velocimetry (PIV) experiments where very small particles are entrained in turbulent flows and optically observed to determine the speed and direction of fluid movement (also known as the velocity field of the fluid). For acceptable tracing accuracy, the particle response time should be faster than the smallest time scale of the flow. Smaller Stokes numbers represent better tracing accuracy; for $\mathrm{Stk} \gg 1$, particles will detach from a flow especially where the flow decelerates abruptly. For $\mathrm{Stk} \ll 1$, particles follow fluid streamlines closely. If $\mathrm{Stk} < 0.1$, tracing accuracy errors are below 1%.[3]

The **Womersley number** (α or Wo) is a dimensionless number in biofluid mechanics and biofluid dynamics. It is a dimensionless expression of the pulsatile flow frequency in relation to viscous effects. It is named after John R. Womersley (1907–1958) for his work with blood flow in arteries.[1] The Womersley number is important in keeping dynamic similarity when scaling an experiment. An example of this is scaling up the vascular system for experimental study. The Womersley number is also important in determining the thickness of the boundary layer to see if entrance effects can be ignored.

The square root of this number is also referred to as **Stokes number**, $\mathrm{Stk} = \sqrt{\mathrm{Wo}}$, due to the pioneering work done by Sir George Stokes on the Stokes second problem.

Pipe flow with oscillating pressure gradient

$u_t = -\frac{p_z}{\rho} + \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$ where $\underline{u} = \left(0,0,u_z(r,t)\right)$ and $u_z = u$

$p_z = \mathrm{Re}\left[\left(\frac{\Delta p}{L}\right)e^{i\omega t}\right]$ Δp pressure fluctuation amplitude between pipe length L ends

Assume $u(r,t) = \mathrm{Re}\left[f(r)e^{i\omega t}\right]$

$$f'' + r^{-1}f' - i\frac{\omega}{\nu}f = \frac{\Delta p}{\mu L} \qquad 0th\ order\ Bessel\ equation$$

$$f(r') = AJ_0\left(i^{3/2}r'\right) + BY_0\left(i^{3/2}r'\right) + i\frac{\Delta p}{\omega\rho L}$$

where $r' = r/\sqrt{\nu/\omega}$

0th Bessel functions of 1st kind complex argument. A and B constants. Since $f(0)$ finite $\Rightarrow B = 0$ ($Y_0(0) \to \infty$).

$$f(r=a) = 0 = AJ_0\left(i^{3/2}\frac{a}{\sqrt{\nu/\omega}}\right) + i\frac{\Delta p}{\omega\rho L}$$

a =radius pipe

$a' = \frac{a}{\sqrt{\nu/\omega}}$

$$A = -\frac{i\Delta p}{\omega\rho L}/J_0\left(i^{3/2}\frac{a}{\sqrt{\nu/\omega}}\right)$$

$$u = \mathrm{Re}\left[\frac{i\,\Delta p}{\omega\rho L}\left(1 - \frac{J_0\left(i^{3/2}r'\right)}{J_0(i^{3/2}a')}\right)e^{i\omega t}\right]$$

Requires special technique Bessel functions arbitrary points complex plane.

$\omega \to 0$ $\quad u(r) = \frac{a^2 - r^2}{4\mu}\frac{dp}{dz}$ steady Poiseuille flow

$\omega \to \infty$ = profile similar Stokes 2nd prob

Exercise 9.32. a) When z is complex, the small-argument expansion of the zeroth-order Bessel function $J_o(z) = 1 - \frac{1}{4}z^2 + ...$ remains valid. Use this to show that (9.43) reduces to (9.6) as $\omega \to 0$ when $dp/dz = \Delta p/L$. The next term in the series is $\frac{1}{64}z^4$. At what value of $a/\sqrt{\nu/\omega}$ is the magnitude of this term equal to 5% of the second term.
b) When z is complex, the large-argument expansion of the zeroth-order Bessel function $J_o(z) \cong (2/\pi z)^{1/2}\cos\left[z - \frac{1}{4}\pi\right]$ remains valid for $|\arg(z)| < \pi$. Use this to show that (9.43) reduces to the velocity profile of a viscous boundary layer on a plane wall beneath an oscillating flow as $\omega \to \infty$:

$$u_z(y,t) = -\frac{\Delta p}{\rho\omega L}\left[\sin(\omega t) - \exp\left\{-y\sqrt{\frac{\omega}{2\nu}}\right\}\sin\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right)\right],$$

where y is the distance from the tube wall, $R = a - y$, $y << a$, and $dp/dz = \Delta p/L$.

Solution 9.32. a) Start from (9.43):

$$u_z(R,t) = \mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[1 - J_o\left(\frac{i^{3/2}R}{\sqrt{\nu/\omega}}\right)\Big/ J_o\left(\frac{i^{3/2}a}{\sqrt{\nu/\omega}}\right)\right]e^{i\omega t}\right\}, \text{ and}$$

use the small argument form of J_o for the limit $\omega \to 0$:

$$\lim_{\omega\to 0} u_z(R,t) = \lim_{\omega\to 0}\mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[1 - \frac{1 - \frac{1}{4}\left(\frac{i^{3/2}R}{\sqrt{\nu/\omega}}\right)^2 + ...}{1 - \frac{1}{4}\left(\frac{i^{3/2}a}{\sqrt{\nu/\omega}}\right)^2 + ...}\right]e^{i\omega t}\right\} = \lim_{\omega\to 0}\mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[1 - \frac{1 + \frac{i\omega R^2}{4\nu} + ...}{1 + \frac{i\omega a^2}{4\nu} + ...}\right]e^{i\omega t}\right\}.$$

Continue simplifying:

$$\lim_{\omega\to 0} u_z(R,t) = \lim_{\omega\to 0}\mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[1 - \left(1 + \frac{i\omega R^2}{4\nu} - \frac{i\omega a^2}{4\nu} + ...\right)\right]e^{i\omega t}\right\}$$

$$= \lim_{\omega\to 0}\mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[-\frac{i\omega R^2}{4\nu} + \frac{i\omega a^2}{4\nu}\right]e^{i\omega t}\right\}$$

$$= \mathrm{Re}\left\{\frac{\Delta p}{\rho L}\left[+\frac{R^2}{4\nu} - \frac{a^2}{4\nu}\right]\right\} = \frac{1}{4\mu}\left(-\frac{\Delta p}{L}\right)\left(a^2 - R^2\right)$$

and this is the same as (9.6) when the pressure gradient is $\Delta p/L$.

To determine when $\frac{1}{64}z^4$ is 5% of $\frac{1}{4}z^2$, set $(0.05)\frac{1}{4}z^2 = \frac{1}{64}z^4$ and determine z. The solution is $|z| = a/\sqrt{\nu/\omega} = \sqrt{0.05(64)/4} = 0.894$.

b) Here, $z = \frac{i^{3/2}R}{\sqrt{\nu/\omega}} = \left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}$, so

$$\cos\left(z-\frac{\pi}{4}\right)=\frac{1}{2}\left[\exp\left\{i\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}-i\frac{\pi}{4}\right\}+\exp\left\{-i\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}+i\frac{\pi}{4}\right\}\right]$$
$$=\frac{1}{2}\left[\exp\left\{\left(-\frac{i}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}-i\frac{\pi}{4}\right\}+\exp\left\{\left(\frac{i}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}+i\frac{\pi}{4}\right\}\right]$$

When $\omega\to\infty$, the first term becomes exponentially small, so

$$\cos\left(z-\frac{\pi}{4}\right)\cong\frac{1}{2}\exp\left\{\left(\frac{i}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}+i\frac{\pi}{4}\right\}\ \text{ as }\ \omega\to\infty.$$

Now use $R=a-y$ in the above expression and collect like factors:

$$\cos\left(z-\frac{\pi}{4}\right)\cong\frac{1}{2}\exp\left\{i\left(\frac{a-y}{\sqrt{2\nu/\omega}}+\frac{\pi}{4}\right)+\frac{a-y}{\sqrt{2\nu/\omega}}\right\}\ \text{ as }\ \omega\to\infty.$$

or:

$$\cos\left(z-\frac{\pi}{4}\right)\cong\frac{e^{i\pi/4}}{2}\exp\left(\frac{a(1+i)}{\sqrt{2\nu/\omega}}\right)\exp\left(\frac{-(1+i)}{\sqrt{2\nu/\omega}}y\right)\ \text{ as }\ \omega\to\infty.$$

So, in this limit:

$$\frac{J_o\left(\frac{i^{3/2}R}{\sqrt{\nu/\omega}}\right)}{J_o\left(\frac{i^{3/2}a}{\sqrt{\nu/\omega}}\right)}=\frac{\sqrt{\frac{2}{\pi}\frac{\sqrt{\nu/\omega}}{i^{3/2}(a-y)}}\frac{e^{i\pi/4}}{2}\exp\left(\frac{a(1+i)}{\sqrt{2\nu/\omega}}\right)\exp\left(\frac{-(1+i)}{\sqrt{2\nu/\omega}}y\right)}{\sqrt{\frac{2}{\pi}\frac{\sqrt{\nu/\omega}}{i^{3/2}a}}\frac{e^{i\pi/4}}{2}\exp\left(\frac{a(1+i)}{\sqrt{2\nu/\omega}}\right)}$$
$$=\sqrt{\frac{a}{a-y}}\exp\left(\frac{-(1+i)}{\sqrt{2\nu/\omega}}y\right)$$
$$\cong\exp\left(\frac{-(1+i)}{\sqrt{2\nu/\omega}}y\right)$$

where the final approximate equality holds when $y \ll a$. Now substitute this approximate ratio of Bessel functions into (9.43) to find:

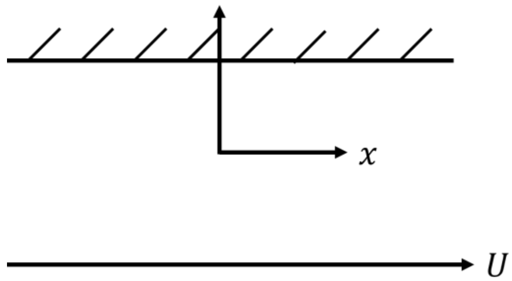
$$u_z(y,t)=\mathrm{Re}\left\{\frac{i\Delta p}{\omega\rho L}\left[1-\exp\left(\frac{-(1+i)y}{\sqrt{2\nu/\omega}}\right)\right]e^{i\omega t}\right\}=\mathrm{Re}\left\{\frac{i\Delta p}{\omega\rho L}\left[e^{i\omega t}-\exp\left(\frac{-y}{\sqrt{2\nu/\omega}}\right)\exp\left(i\omega t-\frac{iy}{\sqrt{2\nu/\omega}}\right)\right]\right\}.$$

Take the real part to reach:

$$u_z(y,t)=-\frac{\Delta p}{\omega\rho L}\left[\sin(\omega t)-\exp\left(\frac{-y}{\sqrt{2\nu/\omega}}\right)\sin\left(\omega t-\frac{y}{\sqrt{2\nu/\omega}}\right)\right],$$

and this is the desired result.

Unsteady flow between two infinite planes



$$u(0,t) = U \qquad \textit{sudden acceleration}$$
$$u(h,t) = 0$$

⇒ find solution $u(y,\infty) \to U_0\left(1-\frac{y}{h}\right)$ i.e., linear Couette flow

Solution can be obtained using a Fourier series technique.

$$\frac{u}{U_0} = \underbrace{\left(1-\frac{y}{h}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp(-n^2\pi^2 t^*)\sin\left(\frac{n\pi y}{h}\right)}_{of\ form\ u=u(y,\infty)-u}$$

$$t^* = \frac{\nu t}{h^2}$$

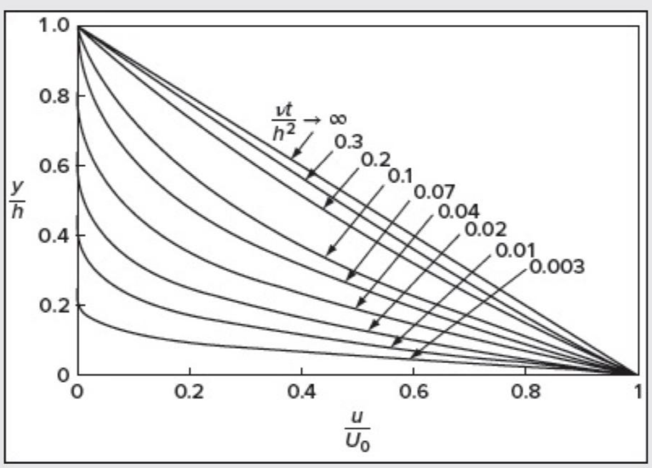


FIGURE 3-23

The development of plane Couette flow due to a suddenly accelerated lower wall