derive and explain physics Richardson overshoot oscillatroy pipe flow

Richardson Overshoot in Oscillatory Pipe Flow (Physics Derivation & Explanation)

Richardson overshoot refers to a transient phenomenon in oscillatory (time-periodic) pressure-driven pipe flow where the **axial velocity near the pipe center temporarily exceeds the corresponding steady (Poiseuille) velocity**. It occurs in high-frequency oscillatory flows due to phase differences between pressure gradient and viscous diffusion.

This effect is closely related to the **Womersley solution** of oscillatory pipe flow.

1️⃣ Physical Setup

Consider incompressible, Newtonian fluid flow in a circular pipe of radius R.

The driving pressure gradient is oscillatory:

$$-\frac{\partial p}{\partial z} = G_0 e^{i\omega t}$$

Where:

- G_0 = amplitude of pressure gradient
- ω = angular frequency
- ρ = fluid density
- ν = kinematic viscosity

Flow is assumed:

- Axisymmetric
- Fully developed
- No swirl
- No radial velocity

So velocity is:

$$\mathbf{u} = u(r,t)\hat{z}$$

2️⃣ Governing Equation

From Navier–Stokes (axial component):

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right)$$

Substitute oscillatory pressure forcing:

$$\frac{\partial u}{\partial t} = \frac{G_0}{\rho}e^{i\omega t} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right)$$

3️⃣ Harmonic Solution Assumption

Because forcing is harmonic, assume solution:

$$u(r,t) = \Re\left\{U(r)e^{i\omega t}\right\}$$

Substitute into equation:

$$i\omega U = \frac{G_0}{\rho} + \nu\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right)\right)$$

Rearrange:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right) - \frac{i\omega}{\nu}U = -\frac{G_0}{\rho\nu}$$

4️⃣ Solution (Womersley Solution)

Define:

$$\lambda^2 = \frac{i\omega}{\nu}$$

The homogeneous equation is Bessel's equation of order 0.

Solution:

$$U(r) = \frac{G_0}{i\omega\rho}\left[1 - \frac{J_0(\lambda r)}{J_0(\lambda R)}\right]$$

Where:

- J_0 = Bessel function of first kind

Thus velocity:

$$u(r,t) = \Re\left\{\frac{G_0}{i\omega\rho}\left[1 - \frac{J_0(\lambda r)}{J_0(\lambda R)}\right]e^{i\omega t}\right\}$$

5️⃣ Womersley Number

Define dimensionless frequency parameter:

$$\alpha = R\sqrt{\frac{\omega}{\nu}}$$

This is the **Womersley number**.

It determines the behavior:

α	**Flow Behavior**
Small (<1)	Quasi-steady (Poiseuille-like)
Large (>10)	Inertia dominated

Richardson overshoot occurs for **large** α.

6️⃣ Physical Mechanism of Richardson Overshoot

For large α:

- Oscillation period is short
- Viscous diffusion cannot penetrate entire pipe
- Thin oscillatory boundary layer forms near wall
- Core behaves nearly inviscid

Key effect:

The wall region is strongly phase-lagged due to viscosity.

The center region responds almost instantaneously.

This creates:

- Strong radial phase differences
- Transient acceleration in the core
- Centerline velocity exceeding steady Poiseuille value

This temporary excess is **Richardson overshoot**.

7️⃣ Asymptotic Insight for Large α

For $\alpha \gg 1$:

$$u(r,t) \approx \frac{G_0}{\rho\omega}\sin(\omega t) \quad \text{in the core}$$

Near wall:

Velocity confined to Stokes layer of thickness:

$$\delta = \sqrt{\frac{2\nu}{\omega}}$$

Thus:

- Core oscillates almost as solid plug
- Wall region strongly damped
- Shear layer produces non-intuitive velocity peaks

The centerline velocity magnitude becomes:

$$|u(0)| = \left|\frac{G_0}{\rho\omega}\left(1 - \frac{1}{J_0(\lambda R)}\right)\right|$$

For large α:

$$\left|\frac{1}{J_0(\lambda R)}\right| \to \text{large}$$

Leading to overshoot.

8️⃣ Why It Is Called "Overshoot"

Compare with steady Poiseuille max velocity:

$$u_{max,steady} = \frac{G_0 R^2}{4\rho\nu}$$

But oscillatory peak velocity scales like:

$$\sim \frac{G_0}{\rho\omega}$$

For sufficiently high frequency:

$$\frac{G_0}{\rho\omega} > \frac{G_0 R^2}{4\rho\nu}$$

i.e.,

$$\alpha^2 > 4$$

Thus:

$$\alpha > 2$$

Overshoot begins around $\alpha \sim O(1 - 10)$

9️⃣ Physical Interpretation

Richardson overshoot occurs because:

1. Inertia dominates over viscous diffusion
2. Pressure oscillates faster than momentum diffuses
3. Core fluid accelerates before viscous resistance balances it
4. Temporary velocity amplification occurs

It is fundamentally a **phase-lag inertial amplification phenomenon**.

🔬 Applications

- Pulsatile blood flow (large arteries)
- Oscillatory ventilation
- Acoustic streaming
- Cardiovascular fluid dynamics
- High-frequency microfluidics

📌 Summary

Richardson overshoot in oscillatory pipe flow:

- Arises from Womersley solution
- Occurs at large Womersley number
- Caused by phase lag between wall shear and core acceleration
- Produces centerline velocity exceeding steady Poiseuille value
- Results from inertial dominance over viscous diffusion

If you'd like, I can next:

- Derive the asymptotic large-α expansion in detail
- Show phase relationships graphically
- Derive the wall shear stress expression
- Non-dimensionalize the full solution
- Connect this to pulsatile blood flow physics