Frame Invariance of Thermodynamic Variables and Viscous Stress in the Navier-Stokes Equations

Core Statement

When transforming the Navier-Stokes equations from an inertial frame to a non-inertial (accelerating/rotating) frame:

- Thermodynamic variables (density ρ, pressure p, temperature T, internal energy e) are **frame-invariant scalars**.

- The net viscous stress tensor $\boldsymbol{\sigma}$ is **frame-invariant**.

Only the acceleration terms in the momentum equation change, introducing fictitious forces.

1. Why Thermodynamic Variables are Frame-Invariant

Thermodynamic variables are **intensive state variables**. They describe the local molecular state of a fluid element and are measured relative to the element's own co-moving frame.

- **Density** ρ: Mass per unit volume of the element. Independent of observer motion.

- **Pressure** p: Momentum flux due to molecular motion *relative to the element*.

- **Temperature** T: Average molecular kinetic energy in the element's rest frame.

Analogy: A pressure gauge attached to a fluid parcel reads the same value whether the parcel is stationary, moving at constant velocity, or accelerating. These properties are intrinsic to the material, not to the observer's state of motion.

$$\rho_R = \rho_I, \quad p_R = p_I, \quad T_R = T_I, \quad e_R = e_I$$

where subscripts R and I denote values in the rotating/non-inertial and inertial frames, respectively.

2. Why the Viscous Stress Tensor is Frame-Invariant

Viscous stress arises from *relative motion within the fluid*, not from its absolute motion. For a Newtonian fluid:

$$\boldsymbol{\sigma} = \lambda(\nabla \cdot \vec{v})\mathbf{I} + 2\mu\mathbf{S}$$

where $\mathbf{S}$ is the **strain-rate tensor**:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

The proof of invariance lies in the transformation of $\mathbf{S}$.

Velocity Transformation

Let $\vec{v}_I$ be velocity in an inertial frame. In a non-inertial frame rotating with angular velocity $\vec{\Omega}(t)$ and with translational origin acceleration $\vec{a}_0(t)$, the velocity is:

$$\vec{v}_R = \vec{v}_I - \vec{\Omega} \times \vec{r} - \vec{V}_0(t)$$

where $\vec{V}_0(t)$ is the translational velocity of the frame's origin.

Gradient Transformation

Compute the velocity gradient in the rotating coordinates:

$$\frac{\partial v_{R,i}}{\partial x_j} = \frac{\partial v_{I,i}}{\partial x_j} - \epsilon_{ijk}\Omega_k$$

The term $-\epsilon_{ijk}\Omega_k$ comes from $\nabla(\vec{\Omega} \times \vec{r})$ and is *anti-symmetric*.

Strain-Rate Tensor Transformation

Form the symmetric part:

$$S_{ij}^{(R)} = \frac{1}{2}\left(\frac{\partial v_{R,i}}{\partial x_j} + \frac{\partial v_{R,j}}{\partial x_i}\right)$$

$$= \frac{1}{2}\left(\frac{\partial v_{I,i}}{\partial x_j} - \epsilon_{ijk}\Omega_k + \frac{\partial v_{I,j}}{\partial x_i} - \epsilon_{jik}\Omega_k\right)$$

Using the anti-symmetry of the Levi-Civita symbol, $\epsilon_{jik} = -\epsilon_{ijk}$:

$$-\epsilon_{ijk}\Omega_k - \epsilon_{jik}\Omega_k = -\epsilon_{ijk}\Omega_k + \epsilon_{ijk}\Omega_k = 0$$

Thus:

$$\boxed{S_{ij}^{(R)} = \frac{1}{2}\left(\frac{\partial v_{I,i}}{\partial x_j} + \frac{\partial v_{I,j}}{\partial x_i}\right) = S_{ij}^{(I)}}$$

The strain-rate tensor is invariant. Since the divergence $\nabla \cdot \vec{v}$ is also invariant (readily shown), the viscous stress $\boldsymbol{\sigma}$ is invariant:

$$\boxed{\boldsymbol{\sigma}^{(R)} = \boldsymbol{\sigma}^{(I)}}$$

3. Transformation of the Navier-Stokes Equations

Only the **material derivative** of velocity (the acceleration) changes. Starting from the inertial-frame momentum equation:

$$\rho\left(\frac{\partial \vec{v}_I}{\partial t} + (\vec{v}_I \cdot \nabla)\vec{v}_I\right) = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \vec{f}_{\text{body}}$$

Transforming to the non-inertial frame introduces fictitious forces:

$$\rho \left[\frac{\partial \vec{v}_R}{\partial t} + (\vec{v}_R \cdot \nabla)\vec{v}_R \right] = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \vec{f}_{\text{body}} + \rho \left[-\vec{a}_0 - 2\vec{\Omega} \times \vec{v}_R - \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) - \frac{d\vec{\Omega}}{dt} \times \vec{r} \right]$$

Where the additional terms are:

- $-\rho\vec{a}_0$: Translational acceleration

- $-2\rho\vec{\Omega} \times \vec{v}_R$: Coriolis force

- $-\rho\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$: Centrifugal force

- $-\rho\frac{d\vec{\Omega}}{dt} \times \vec{r}$: Euler force

Crucially, p and $\boldsymbol{\sigma}$ remain unchanged in form and value.

Summary: Physical Interpretation

- **Thermodynamics** describes the intrinsic *state* of matter.

- **Viscous stress** describes internal friction due to *relative deformation* (velocity gradients), not due to bulk translation or rotation.

Frame changes affect kinematic acceleration (inertial effects) but not the local constitutive physics governing pressure and viscous stress, which are determined in the fluid element's co-moving frame.