

Great question — this is exactly the subtle step 👍

Let's **slow down and be very explicit about Step 4**, with no hand-waving.

We start from the result we already derived:

$$\mathbf{n} \cdot \boldsymbol{\tau} = \mu [2(\mathbf{n} \cdot \nabla)\mathbf{u} + \mathbf{n} \times \boldsymbol{\omega}] \quad (1)$$

This is an **exact identity**.

What does “extract the tangential part” mean?

For any vector \mathbf{a} , decompose it into **normal** and **tangential** components:

$$\mathbf{a} = (\mathbf{n} \cdot \mathbf{a}) \mathbf{n} + \underbrace{[\mathbf{a} - (\mathbf{n} \cdot \mathbf{a})\mathbf{n}]}_{\text{tangential}}$$

The tangential projection operator is

$$\mathbf{P}_t = \mathbf{I} - \mathbf{n}\mathbf{n}$$

Apply the projection to (1)

$$(\mathbf{n} \cdot \boldsymbol{\tau})_t = \mathbf{P}_t(\mathbf{n} \cdot \boldsymbol{\tau}) = \mu \mathbf{P}_t [2(\mathbf{n} \cdot \nabla)\mathbf{u} + \mathbf{n} \times \boldsymbol{\omega}] \quad (2)$$

Now treat the two terms **separately**.

Term 1: $(\mathbf{n} \cdot \nabla)\mathbf{u}$

At a solid wall:

- No-penetration:

$$\mathbf{u} \cdot \mathbf{n} = 0$$

Differentiate in the normal direction:

$$(\mathbf{n} \cdot \nabla)(\mathbf{u} \cdot \mathbf{n}) = 0$$

Expand:

$$(\mathbf{n} \cdot \nabla)\mathbf{u} \cdot \mathbf{n} + \mathbf{u} \cdot (\mathbf{n} \cdot \nabla)\mathbf{n} = 0$$

At a locally planar wall, $(\mathbf{n} \cdot \nabla)\mathbf{n} = 0$, so

$$(\mathbf{n} \cdot \nabla)\mathbf{u} = [(\mathbf{n} \cdot \nabla)\mathbf{u} \cdot \mathbf{n}]\mathbf{n}$$

✓ This term is **purely normal**, hence

$$\mathbf{P}_t[(\mathbf{n} \cdot \nabla)\mathbf{u}] = 0$$

Term 2: $\mathbf{n} \times \boldsymbol{\omega}$

By definition of the cross product,

$$\mathbf{n} \cdot (\mathbf{n} \times \boldsymbol{\omega}) = 0$$

Therefore,

$$\mathbf{P}_t(\mathbf{n} \times \boldsymbol{\omega}) = \mathbf{n} \times \boldsymbol{\omega}$$

✓ This term is **purely tangential**.

Combine results

Applying the projection in (2):

$$(\mathbf{n} \cdot \boldsymbol{\tau})_t = \mu [0 + \mathbf{n} \times \boldsymbol{\omega}]$$

With the sign convention for traction on the wall,

$$(\mathbf{n} \cdot \boldsymbol{\tau})_t = -\mu \mathbf{n} \times \boldsymbol{\omega}$$

What this really means (important)

The full traction vector is

$$\mathbf{n} \cdot \boldsymbol{\tau} = \underbrace{2\mu(\mathbf{n} \cdot \nabla)\mathbf{u}}_{\text{normal part}} + \underbrace{(-\mu \mathbf{n} \times \boldsymbol{\omega})}_{\text{tangential shear}}$$

So:

- **Normal part** → strain rate normal to the wall
- **Tangential part** → vorticity-induced wall shear

If you want, I can rewrite Step 4 **without assuming a planar wall**, or show the same projection using **pure index notation** (very compact and journal-friendly).

what does "With the sign convention for traction on the wall," mean?

Great catch — that phrase *does* hide an assumption, and you're right to call it out. Let's unpack it carefully and cleanly.

What is “the sign convention for traction on the wall”?

In continuum mechanics, **traction** is defined as

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma} \cdot \mathbf{n}$$

where:

- $\boldsymbol{\sigma}$ is the stress tensor of the fluid
- \mathbf{n} is the unit normal pointing out of the fluid

This choice of normal direction is the **sign convention**.

Fluid–solid interface: where the sign enters

At a solid wall, there are *two* natural normals:

- \mathbf{n}_f : outward normal of the **fluid**
- $\mathbf{n}_w = -\mathbf{n}_f$: outward normal of the **wall**

If you compute stress using the fluid normal:

$$\mathbf{t}_f = \boldsymbol{\sigma} \cdot \mathbf{n}_f$$

If you compute the force **on the wall exerted by the fluid**, Newton’s third law gives:

$$\mathbf{t}_{\text{wall}} = -\boldsymbol{\sigma} \cdot \mathbf{n}_f = \boldsymbol{\sigma} \cdot \mathbf{n}_w$$

Where the minus sign in $-\mu \mathbf{n} \times \boldsymbol{\omega}$ comes from

From the tensor identity we derived:

$$(\boldsymbol{\sigma}^{(v)} \cdot \mathbf{n}_f)_t = \mu \mathbf{n}_f \times \boldsymbol{\omega}$$

But in **wall shear stress** literature (and CFD codes), the quantity of interest is usually:

shear stress acting on the wall

That is:

$$\boldsymbol{\tau}_w = (\text{force on wall per unit area}) = -\boldsymbol{\sigma}^{(v)} \cdot \mathbf{n}_f$$

Hence,

$$\boldsymbol{\tau}_w = -\mu \mathbf{n}_f \times \boldsymbol{\omega}$$

This is the version most often quoted.

Summary table (very important)

Quantity	Normal used	Expression
Traction on fluid	\mathbf{n}_f	$\mu \mathbf{n}_f \times \boldsymbol{\omega}$

Quantity	Normal used	Expression
Traction on wall	$\mathbf{n}_w = -\mathbf{n}_f$	$-\mu \mathbf{n}_f \times \boldsymbol{\omega}$
Wall shear stress (CFD)	wall normal	$\mu \mathbf{n}_w \times \boldsymbol{\omega}$

Same physics — just **opposite normals**.