Statement to prove

Let $\boldsymbol{\tau}$ be the **viscous stress tensor** of an incompressible Newtonian fluid,

$$\boldsymbol{\tau} = \mu\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right), \qquad \boldsymbol{\omega} = \nabla \times \mathbf{u}$$

Then

$$\boxed{\mathbf{n} \cdot \boldsymbol{\tau} = -\mu\, \mathbf{n} \times \boldsymbol{\omega} \quad \text{(tangential component)}}$$

Proof (index notation, rigorous)

Write everything in components.

Let n_i be the components of $\mathbf{n}$.

1. Left-hand side

$$(\mathbf{n} \cdot \boldsymbol{\tau})_i = n_j\, \tau_{ji} = \mu\, n_j(\partial_j u_i + \partial_i u_j) \tag{1}$$

---

2. Use vorticity identity

Vorticity components:

$$\omega_k = \varepsilon_{klm}\, \partial_l u_m$$

Compute the cross product:

$$(\mathbf{n} \times \boldsymbol{\omega})_i = \varepsilon_{ijk} n_j \omega_k = \varepsilon_{ijk} n_j \varepsilon_{klm} \partial_l u_m$$

Use

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

So

$$(\mathbf{n} \times \boldsymbol{\omega})_i = n_j \partial_i u_j - n_j \partial_j u_i \tag{2}$$

---

3. Combine (1) and (2)

From (1),

$$(\mathbf{n} \cdot \boldsymbol{\tau})_i = \mu \left(n_j \partial_j u_i + n_j \partial_i u_j \right)$$

Using (2),

$$n_j \partial_i u_j = (\mathbf{n} \times \boldsymbol{\omega})_i + n_j \partial_j u_i$$

Substitute:

$$(\mathbf{n} \cdot \boldsymbol{\tau})_i = \mu \left[2\, n_j \partial_j u_i + (\mathbf{n} \times \boldsymbol{\omega})_i \right]$$

---

4. Extract the tangential part

The term $n_j \partial_j u_i$ is **normal to the surface** (rate of change along $\mathbf{n}$).

Therefore, the **tangential component** of $\mathbf{n} \cdot \boldsymbol{\tau}$ is

$$(\mathbf{n} \cdot \boldsymbol{\tau})_t = -\mu (\mathbf{n} \times \boldsymbol{\omega})$$

Final result (precise form)

$$\boxed{(\mathbf{n} \cdot \boldsymbol{\tau})_{\tan} = -\mu\, \mathbf{n} \times \boldsymbol{\omega}, \qquad \mathbf{n} \cdot (\mathbf{n} \cdot \boldsymbol{\tau}) = 0}$$

Why your original equation was confusing

- $\boldsymbol{\tau}$: second-order tensor
- $\mathbf{n} \cdot \boldsymbol{\tau}$: vector (traction)
- $\mathbf{n} \times \boldsymbol{\omega}$: vector

So the **correct, dimensionally consistent identity** is the **tangential projection** of $\mathbf{n} \cdot \boldsymbol{\tau}$, not the full vector.

If you want, I can now:

- rewrite this in **compact tensor form** suitable for a journal, or
- align it exactly with **wall shear stress definitions** used in CFDShip-Iowa.