Rayleigh's Inviscid Instability Argument

To study the stability of inviscid circular flow between concentric cylinders, Lord Rayleigh developed an analogy with the stability of a fluid of variable density under the force of gravity. Taylor summarizes this as: "the varying centrifugal force of the different layers of fluid plays the part of gravity and the resulting condition for stability is that the square of the circulation must increase continuously in passing from the inner to the outer cylinder, just as the density of a fluid under gravity must decrease continuously with height in order that it may be in stable equilibrium."gif

To examine this argument, consider a ring of fluid in a flow field with a velocity distribution $u_\theta(r)$. The angular momentum per unit volume measured about the central axis is given by $\rho r u_\theta$. Expressing the velocity $u_\theta(r)$ in terms of the angular velocity, ω, gives $u_\theta = r\omega$, and therefore

$$\frac{\text{Angular momentum}}{\rho\ \text{Volume}} = r^2\omega \ . \qquad (3)$$

One can also consider the circulation of the fluid along a circular path,

$$\oint \mathbf{u}\cdot d\,\mathbf{x} \ , \qquad (4)$$

and notice that the circulation has the dimensions of angular momentum, $r^2\omega$.

In an inviscid fluid, the angular momentum of a small fluid element is conserved. This can be seen from the Euler equations for inviscid flow,

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p \ . \qquad (5)$$

We assume the motion is axisymmetric, so that the θ-component of the Euler equations becomes

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z} = 0 \ . \qquad (6)$$

Multiplication by r leads to

$$\frac{\partial (ru_\theta)}{\partial t} + u_r\frac{\partial (ru_\theta)}{\partial r} + u_z\frac{\partial (ru_\theta)}{\partial z} = 0 \qquad (7)$$

$$\frac{\partial (ru_\theta)}{\partial t} + \mathbf{u}\cdot\nabla(ru_\theta) = 0 \ . \qquad (8)$$

This is simply the material derivative of ru_θ, so that

$$\frac{D}{Dt}(ru_\theta) = 0 \ , \qquad (9)$$

and we see that ru_θ or $r^2\omega$ is constant and angular momentum is conserved.

Next we consider the perturbation resulting when two rings of fluid at different radii are interchanged while conserving angular momentum. The change in angular momentum is

$$d(r^2\omega) = 2r\omega dr + r^2 d\omega = 0 \ , \qquad (10)$$

from which it follows that the corresponding change in angular velocity for a displaced fluid ring is

$$d\omega = -\frac{2\omega}{r}dr \ . \qquad (11)$$

We now consider the stability of such a perturbation. The rotating fluid is stable if the displaced fluid ring experiences a force which tends to return it to its original position. Thus, we can imagine displacing a small fluid element from r to $r+\delta r$ such that angular momentum is conserved, and then asking whether the pressure in the surrounding fluid is able to maintain equilibrium. We seek a comparison between the pressure in the equilibrium state, and the pressure in the disturbed state. If the pressure in the undisturbed case is greater than in the disturbed state, it will tend to push the disturbed ring of fluid back to its equilibrium position and maintain stability in the flow. If, on the other hand, the pressure of the disturbed ring of fluid is less than the pressure in the undisturbed fluid surrounding it, instability will set in.

The r-component of the Euler equation gives the centrifugal pressure gradient as

$$\frac{\partial p}{\partial r} = \rho\frac{{u_\theta}^2}{r} = \rho r\omega^2 \qquad (12)$$

so that $p = \frac{1}{2}\rho r^2\omega^2$ is the pressure to within a constant. For fluid elements in the equilibrium state, the angular velocity at r is $\omega(r)$, and at $r+\delta r$ it is $\omega(r+\delta r)$, or $\omega(r) + \frac{d\omega}{dr}(\delta r)$ to first order. The pressure in the undisturbed flow at $r+\delta r$ is then

$$p_{undist} = \frac{1}{2}\rho(r+\delta r)^2\left[\omega(r) + \frac{d\omega}{dr}\delta r\right]^2 \ . \qquad (13)$$

In the disturbed flow case, the angular velocity at r is $\omega(r)$ and at $r+\delta r$ it is $\omega(r+\delta r) \approx \omega(r) + d\omega$. Since angular momentum is conserved, $d\omega = -2\frac{\omega(r)}{r}\delta r$ (from Equation [11](#)), so we then have

$$\omega(r+\delta r) \approx \omega(r) - 2\frac{\omega(r)}{r}\delta r \ . \qquad (14)$$

The pressure in the disturbed flow at $r+\delta r$ is therefore given by

$$p_{dist} = \frac{1}{2}\rho(r+\delta r)^2\left[\omega(r) - 2\frac{\omega}{r}\delta r\right]^2 \ . \qquad (15)$$

Taking the difference between Eqs. [13](#) and [15](#) we obtain

$$p_{dist} - p_{undist} = \frac{1}{2}\rho(r+\delta r)^2\left[-4\frac{\omega^2}{r}\delta r - 2\omega\frac{d\omega}{dr}\delta r + O(\delta r^2)\right] \qquad (16)$$

so that

$$p_{dist} - p_{undist} \propto -2\omega\left[2\frac{\omega}{r} + \frac{d\omega}{dr}\right]\delta r = -2\frac{\omega}{r^2}\frac{d(r^2\omega)}{dr}\delta r \ . \qquad (17)$$

Stability requires that $p_{dist} < p_{undist}$, which implies the following conditions:

$$\textbf{Stability:} \quad \frac{d}{dr}(r^2\omega) > 0 \qquad (18)$$

$$\textbf{Instability:} \quad \frac{d}{dr}(r^2\omega) < 0 \qquad (19)$$

If we think of $r^2\omega$ as the circulation of the flow, we come to Rayleigh's conclusion that "A circulation always increasing outwards ... ensures stability."

Returning to the case of Couette flow, we have $u_\theta = Ar + B/r$ so that $\frac{d}{dr}(r^2\omega) = 2A$. Substituting the value given for A in Equation [2](#), we require that for stability

$$\frac{\Omega_1(\eta^2-\mu)}{\eta^2-1} > 0 \ . \qquad (20)$$

Since $\eta^2 < 1$ by definition, we must have $\eta^2 < \mu$ for stable flow, or

$$\Omega_1 {R_1}^2 < \Omega_2 {R_2}^2 \quad \boxed{\text{Rayleigh's Stability Criterion}} \qquad (21)$$

In Ω_1, Ω_2-space, the stable region is therefore bounded by the positive Ω_2-axis and the line $\Omega_2 = \Omega_1\eta^2$. From this we see that if the outer cylinder is fixed ($\Omega_2 = 0$), the flow is always unstable.

When $\mu < 0$ and the cylinders are rotating in opposite directions, instability is predicted for all rotation speeds in the absence of viscosity. As explained by Chandrasekhargif, however, the instability is only partial. It occurs between the interior cylinder and the nodal surface, defined by $\Omega = 0$, which separates the flow into two parts. The radius of this surface is given by

$$R_0 = \eta\left(\frac{1+|\mu|}{\eta^2+|\mu|}\right)^{1/2} R_2 \ , \qquad (22)$$

so there is stability for $R_0 < r < R_2$ and instability for $R_1 < r < R_0$. Thus, according to Taylor, "In the inner region the square of the circulation decreases outwards, so that centrifugal force tends to make the flow unstable. In the outer region the square of the circulation increases so that centrifugal force tends to make the flow stable"gif (p. 69). It is interesting to note that as $\mu \to -\infty$, $R_0 \to R_1$, and the area where the fluid is unstable in the inviscid case becomes infinitesimally small.

When viscous effects are included, there is an added area of stability in Ω_1, Ω_2-space due to viscous stabilization. Stable flow with $\Omega_2 = 0$, for instance, is possible for Ω_1 less than some critical value. In general, flow is unstable only if

$$\Omega_1 {R_1}^2 > \Omega_2 {R_2}^2 \quad \text{and} \quad \Omega_1 > \Omega_{crit} \ . \qquad (23)$$

Taylor was the first to investigate this case theoretically and experimentally, and determine the new criterion for the onset of instability. His experiments confirmed his findings, and showed that the marginal state was stationary and showed a steady axisymmetric cellular patterngif. He found that when the two cylinders rotate in the same direction, Rayleigh's criterion for inviscid flow still holds, with a small added region of stability. When the cylinders rotate in opposite directions, however, Taylor found a large region of stable flow possible, unlike Rayleigh's prediction that all rotation speeds lead to instability.

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Alternate Derivation Rayleigh Inviscid Criterion

Nondimensionalize perturbation equations

$r' = r/r_1 \quad z' = z/r_1 \quad t' = t\omega_1 \qquad$ drop $'$

assume perturbations

$u' = u(r) \cos \lambda z \, e^{\sigma t} \qquad u = u_r$

$v' = v(r) \cos \lambda z \, e^{\sigma t} \qquad v = u_\theta \qquad$ in previous notation

$w' = w(r) \sin \lambda z \, e^{\sigma t} \qquad w = u_z$

eliminate p' from u_r and u_z equations

$$(L - \lambda^2 - \sigma R)(L - \lambda^2) u = 2\lambda^2 R \left(A_1 + \frac{B_1}{r^2}\right) v \qquad (1)$$

and u_θ equation becomes

$$(L - \lambda^2 - \sigma R) v = 2RAu \qquad (2)$$

$L = D^2 + D/r - 1/r^2 \qquad D = \frac{d}{dr}$

$A_1 = A/\omega_1 \quad B_1 = B/r_1^2\omega_1 \quad R = \frac{\omega_1 r_1}{\nu} \qquad U_\theta = V(r) = Ar + B/r$

Basic Solution

Continuity equation: $\lambda r w = -D(ru)$

BC: $u = Du = 0$, $v = 0$ $\quad r = 1$ and $r = \alpha = r_2/r_1 \qquad (3)$

Equations (1), (2) and (3) constitute the stability problem

For $R \to \infty$ $(\nu = 0)$, R terms dominate and divide by R

$$(L - \lambda^2) u = -2\lambda^2 \sigma^{-1} (A_1 + B_1/r^2) v$$

$$-\sigma v = 2 A_1 u$$

$$\Rightarrow \quad (L - \lambda^2) v - \frac{4\lambda^2 A_1}{\sigma^2} \left(A_1 + \frac{B_1}{r^2}\right) v = 0$$

substitute $\omega = A_1 + B_1/r^2$

$$D\left[\frac{1}{r} D(rv)\right] - \lambda^2 \left(\frac{1}{\sigma^2}\frac{F}{r} + \frac{1}{r}\right) rv = 0$$

$$F = \frac{1}{r^3} \frac{d}{dr} \Gamma^2 \qquad \Gamma = \omega r^2$$

BC: $rv = 0$ $r = 1$ and $r = \infty$

Sturm-Liouville System:

$F > 0$ $\Gamma^2 \uparrow$ outward σ^2 real < 0 ie stable

$F < 0$ $\sigma^2 > 0$ one root > 0 ie unstable

$F > 0$ some regions < 0 other σ^2 both > 0 and < 0 eigenvalues ie unstable

only valid $\mu = 0$ — $\frac{d}{dr}\Gamma^2 < 0$ unstable

valid $\mu = 0$ and $\mu \neq 0$ — $\frac{d}{dr}\Gamma^2 > 0$ stable