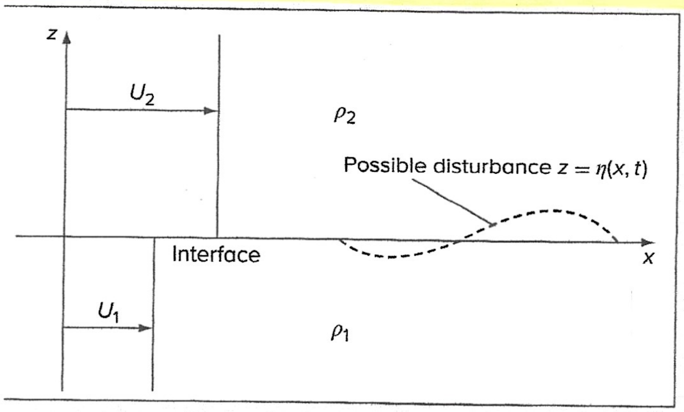
Wind generated waves: Kelvin-Helmholtz Instability



Horizontal interface divides two uniform flows of different U and ρ.

$\rho = $ const, $\mu = 0$ $\underline{\omega} = 0$

FIGURE 5-2
Sketch and nomenclature for the Kelvin–Helmholtz interfacial instability.

Step 1 Basic Flow

$z < 0$: $\phi_1 = U_1 x$ $\quad p_1 = p_0 - \rho_1 g z$

$z > 0$: $\phi_2 = U_2 x$ $\quad p_2 = p_0 - \rho_2 g z$

Both flows have velocity potential and hydrostatic $p(z)$

Since $\mu = 0$ tangential slip at interface: vortex sheet with discontinuity in velocity. Recall Chapter 5b: potential flow vortex sheet and diffusion vortex sheet.

Step 2 Add disturbance

$$\phi_1 = U_1 x + \hat{\phi}_1(x, z, t) \qquad \phi_2 = U_2 x + \hat{\phi}_2(x, z, t)$$

Interface BC:

Dynamic $\Rightarrow$ pressure continuous across interface

unsteady Bernoulli $\quad p_i = \rho_i C_i - \rho_i \frac{\partial \phi_i}{\partial t} - \frac{\rho_i}{2} |\nabla \phi_i|^2 - \rho_i g z$

For basic flow on $z = 0$ $\quad \rho_1 C_1 - \frac{1}{2}\rho_1 U_1^2 = \rho_2 C_2 - \frac{1}{2}\rho_2 U_2^2$

For perturbed flow on $z = \eta$

$$\rho_1 C_1 - \rho_1 \frac{\partial \phi_1}{\partial t} - \frac{\rho_1}{2}|\nabla \phi_1|^2 - \rho_1 g z = \rho_2 C_2 - \rho_2 \frac{\partial \phi_2}{\partial t} - \frac{\rho_2}{2}|\nabla \phi_2|^2 - \rho_2 g z$$

Kinematics $\Rightarrow$ vertical velocities = interfacial motion
or equivalently $F = z - \eta(x,t)$ is a
material surface particles which lie on
surface stay on surface i.e.
surface is stream surface

$$\frac{DF}{Dt} = 0 \quad \text{on } z = \eta$$
$$= \frac{\partial F}{\partial t} + \underline{V_j} \cdot \nabla \eta$$
$$-\eta_t - (U_1 + \hat{\phi}_{1_x})\eta_x + \hat{\phi}_{1_z} = 0$$
$$-\eta_t - (U_2 + \hat{\phi}_{2_x})\eta_x + \hat{\phi}_{2_z} = 0$$

$$\underline{V_1} = (U_1 + \hat{\phi}_{1_x})\hat{i} + \hat{\phi}_{1_z}\hat{k}$$
$$\underline{V_2} = (U_2 + \hat{\phi}_{2_x})\hat{i} + \hat{\phi}_{2_z}\hat{k}$$
$$\nabla F = F_x \hat{i} + F_z \hat{k}$$
$$= -\eta_x \hat{i} + \hat{k}$$

Far field BC: disturbances $\rightarrow 0$ for large z

$$\nabla \hat{\phi}_2 \rightarrow 0 \quad z \rightarrow \infty \qquad \nabla \hat{\phi}_1 \rightarrow 0 \quad z \rightarrow -\infty$$

Step 3
Subtract
basic
flow from
disturbance
equation

$$\nabla^2 \phi_1 = \nabla^2 \hat{\phi}_1 = 0 \qquad \nabla^2 \phi_2 = \nabla^2 \hat{\phi}_2 = 0$$

Both basic & disturbance equations
satisfy Laplace equation

Step 4
linearize
equations

Disturbances are $<<$ basic flow; therefore,
neglect products of disturbances &
apply BC on $z = 0$

$$\rho_1 c_1 - \rho_1 \hat{\phi}_{1_t} - \frac{\rho_1}{2}\left[(U_1 + \hat{\phi}_{1_x})^2 + \hat{\phi}_{1_z}^2\right] - \rho_1 g \eta =$$
$$\rho_2 c_2 - \rho_2 \hat{\phi}_{2_t} - \frac{\rho_2}{2}\left[(U_2 + \hat{\phi}_{2_x})^2 + \hat{\phi}_{2_z}^2\right] - \rho_2 g \eta \qquad \text{on } z = 0$$

$$\rho_1 c_1 - \rho_2 c_2 = \tfrac{1}{2}\rho_1 U_1^2 - \tfrac{1}{2}\rho_1 U_2^2 \qquad \text{on } z = 0$$

$$\rho_1\left[\hat{\phi}_{1_t} + U_1 \hat{\phi}_{1_x} + g\eta\right] = \rho_2\left[\hat{\phi}_{2_t} + U_2 \hat{\phi}_{2_x} + g\eta\right] \qquad \text{on } z = 0$$

$$\eta_t = -U_1 \eta_x + \hat{\phi}_{1_z} = -U_2 \eta_x + \hat{\phi}_{2_z} \qquad \text{on } z = 0$$

Step 5 Assume normal mode disturbance (Similar separation of variables)

$$\hat{\phi}_j = A_j(z)\, e^{i(\alpha x - \sigma t)}$$
$$\hat{\phi}_1 = A_1(z)\, e^{i(\alpha x - \sigma t)} \qquad \hat{\phi}_2 = A_2(z)\, e^{i(\alpha x - \sigma t)}$$

α = wave number

σ = temporal growth rate $= \sigma_r + i\sigma_i$

$-i\sigma = -i\sigma_r + \sigma_i \quad \Rightarrow \sigma_i > 0$ unstable

Step 6 Solve eigenvalue problem

$$\nabla^2 \hat{\phi}_1 = 0 = \hat{\phi}_{1_{xx}} + \hat{\phi}_{1_{zz}}$$

$$\hat{\phi}_{1_x} = A_1 i\alpha\, e^{i(\alpha x - \sigma t)}$$
$$\hat{\phi}_{1_{xx}} = -A_1 \alpha^2 e^{i(\alpha x - \sigma t)}$$
$$\hat{\phi}_{1_{zz}} = A_{1_{zz}} e^{i(\alpha x - \sigma t)}$$

$$-\alpha^2 A_1 + A_{1_{zz}} = 0$$
$$-\alpha^2 A_2 + A_{2_{zz}} = 0$$

Exponential solutions $A_j = A_j' e^{\pm\alpha z}$
where $\pm$ determined bound for field BC

$$\hat{\phi}_1 = A_1' e^{[i(\alpha x - \sigma t) + \alpha z]} \qquad \hat{\phi}_2 = A_2' e^{[i(\alpha x - \sigma t) - \alpha z]}$$

Also assume $\eta = \eta_0 e^{i(\alpha x - \sigma t)}$

$$\eta_x = \eta_0 (i\alpha)\, e^{i(\alpha x - \sigma t)} \qquad \eta_t = \eta_0 (-i\sigma)\, e^{i(\alpha x - \sigma t)}$$

$$\hat{\Phi}_{1z} = A_1' \alpha\, e^{i(\alpha x - \sigma t)} e^{\alpha z} \qquad \hat{\Phi}_{1x} = A_1' (i\alpha)\, e^{i(\alpha x - \sigma t)} e^{\alpha z}$$

$$\hat{\Phi}_{2z} = A_2' (-\alpha)\, e^{i(\alpha x - \sigma t)} e^{-\alpha z} \qquad \hat{\Phi}_{2x} = A_2' (i\alpha)\, e^{i(\alpha x - \sigma t)} e^{-\alpha z}$$

$$\hat{\Phi}_{1t} = A_1' (-i\sigma)\, e^{i(\alpha x - \sigma t)} e^{-\alpha z} \qquad \hat{\Phi}_{2t} = A_2' (-i\sigma)\, e^{i(\alpha x - \sigma t)} e^{-\alpha z}$$

$z = 0$

$$-U_1 \eta_0 (i\alpha) + \alpha A_1' = \eta_0 (-i\sigma) = -U_2 \eta_0 (i\alpha) - \alpha A_2'$$

$$\alpha A_1' = i\eta_0 (\alpha U_1 - \sigma) \qquad A_1' = i\eta_0 (U_1 - \sigma/\alpha)$$

$$\alpha A_2' = -i\eta_0 (\alpha U_2 - \sigma) \qquad A_2' = -i\eta_0 (U_2 - \sigma/\alpha)$$

$z = 0$

$$\rho_1 \left[A_1'(-i\sigma) + U_1 A_1' (i\alpha) + g\eta_0 \right] = \rho_2 \left[A_2'(-i\sigma) + U_2 A_2'(i\alpha) + g\eta_0 \right]$$

$$\rho_1 \left[A_1'(-i\sigma) - U_1 \eta_0 (\alpha U_1 - \sigma) + g\eta_0 \right] = \rho_2 \left[A_2'(-i\sigma) + U_2 \eta_0 (\alpha U_2 - \sigma) + g\eta_0 \right]$$

$$\rho_1 \left[i\cancel{\eta_0}(U_1 - \sigma/\alpha)(-i\sigma) - U_1 \cancel{\eta_0} (\alpha U_1 - \sigma) + g\cancel{\eta_0} \right] =$$

$$\rho_2 \left[-i\cancel{\eta_0}(U_2 - \sigma/\alpha)(-i\sigma) + U_2 \cancel{\eta_0} (\alpha U_2 - \sigma) + g\cancel{\eta_0} \right]$$

$$\rho_1 \left[-\left(U_1 - \frac{\sigma}{\alpha}\right)\sigma - U_1 (\alpha U_1 - \sigma) + g \right] =$$

$$\rho_2 \left[-\left(U_2 - \frac{\sigma}{\alpha}\right)\sigma + U_2 (\alpha U_2 - \sigma) + g \right]$$

$$\rho_1 \left[(U_1 - \sigma/\alpha)(\sigma - U_1 \alpha) + g \right] = =$$

$$\rho_2 \left[-(U_2 - \sigma/\alpha)(\sigma - U_2 \alpha) + g \right]$$

$$\rho_1 \left[(\alpha U_1 - \sigma)(\sigma - U_1 \alpha) + g\alpha \right] = \rho_2 \left[-(\alpha U_2 - \sigma)(\sigma - U_2 \alpha) + g\alpha \right]$$

$$\alpha v_1 \sigma - \alpha^2 v_1^2 - \sigma^2 + \alpha v_1 \sigma = -(\sigma - \alpha v_1)^2 = -\sigma^2 + 2\alpha v_1 \sigma - \alpha^2 v_1^2$$

$$-(\alpha v_2 - \sigma)(\sigma - v_2 \alpha) = (\sigma - \alpha v_2)^2 = \sigma^2 - 2\alpha v_2 \sigma + \alpha^2 v_2^2$$

$$\rho_1 \left[-(\sigma - \alpha v_1)^2 + g\alpha \right] = \rho_2 \left[(\sigma - \alpha v_2)^2 + g\alpha \right]$$

$$-\rho_1 \sigma^2 + 2\rho_1 \alpha v_1 \sigma - \rho_1 \alpha^2 v_1^2 + \rho_1 g \alpha$$
$$-\rho_2 \sigma^2 + 2\rho_2 \alpha v_2 \sigma - \rho_2 \alpha^2 v_2^2 - \rho_2 g \alpha = 0$$

$$-\sigma^2(\rho_1 + \rho_2) + 2\alpha\sigma(\rho_1 v_1 + \rho_2 v_2) - \alpha^2(\rho_1 v_1^2 + \rho_2 v_2^2) + g\alpha(\rho_1 - \rho_2) = 0$$

$$\sigma^2 + \sigma\left[\frac{-2\alpha(\rho_1 v_1 + \rho_2 v_2)}{(\rho_1 + \rho_2)}\right] + \frac{\alpha^2}{(\rho_1 + \rho_2)}\left(\rho_1 v_1^2 + \rho_2 v_2^2 - \frac{g}{\alpha}(\rho_1 - \rho_2)\right) = 0$$

$$ax^2 + bx + c = 0 \qquad x = (-b \pm \sqrt{b^2 - 4ac})/2a$$

$$a = 1 \qquad b = -2\alpha(\rho_1 v_1 + \rho_2 v_2)/(\rho_1 + \rho_2)$$

$$c = \frac{\alpha^2}{(\rho_1 + \rho_2)}\left[\rho_1 v_1^2 + \rho_2 v_2^2\right] - \alpha g \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$$

$$b^2 = 4\alpha^2(\rho_1 v_1 + \rho_2 v_2)^2/(\rho_1 + \rho_2)^2$$

$$-4ac = -\frac{4\alpha^2}{(\rho_1 + \rho_2)}(\rho_1 v_1^2 + \rho_2 v_2^2) + 4\alpha g \frac{(\rho_1 - \rho_2)}{(\rho_1 + \rho_2)}$$

$$(\rho_1 + \rho_2)^{-2}\left[(\rho_1 v_1 + \rho_2 v_2)^2 - (\rho_1 v_1^2 + \rho_2 v_2^2)(\rho_1 + \rho_2)\right]$$

$$\cancel{\rho_1^2 v_1^2} + 2\rho_1\rho_2 v_1 v_2 + \cancel{\rho_2^2 v_2^2} - \cancel{\rho_1^2 v_1^2} - \rho_1\rho_2 v_2^2$$
$$- \rho_2\rho_1 v_1^2 - \cancel{\rho_2^2 v_2^2}$$

$$\rho_1\rho_2(2v_1 v_2 - v_2^2 - v_1^2) = -\rho_1\rho_2(v_1 - v_2)^2$$

$$\sigma = \frac{\alpha(\rho_1 U_1 + \rho_2 U_2)}{\rho_1+\rho_2} \pm \left[\overset{(1)}{\frac{\alpha g(\rho_1-\rho_2)}{(\rho_1+\rho_2)}} - \overset{(2)}{\frac{\alpha^2 \rho_1\rho_2(U_1-U_2)^2}{(\rho_1+\rho_2)^2}}\right]^{1/2}$$

$= \sigma_r + i\sigma_i$ $\quad \sigma_i > 0$ unstable $\quad \sqrt{-p} = i\sqrt{p}$ $\quad -p<0$

For (2) < (1) $\quad \sigma_i = 0$ ∴ neutral stability

For (1) < (2) $\quad \sigma_i$ is + root $\sigma_i > 0$.

$$\frac{\alpha g(\rho_1-\rho_2)}{(\rho_1+\rho_2)} < \frac{\alpha^2\rho_1\rho_2(U_1-U_2)^2}{(\rho_1+\rho_2)^2}$$

or $\quad g(\rho_1-\rho_2)(\rho_1+\rho_2) < \alpha\rho_1\rho_2(U_1-U_2)^2$

or $\quad g(\rho_1^2-\rho_2^2) < \alpha\rho_1\rho_2(U_1-U_2)^2$

∴ if ΔU large enough or $(\rho_1^2-\rho_2^2)$ small enough or α large enough flow unstable (ie λ small) ⇒ vortex sheets always unstable

Step 7 not needed as analytical solution provides stability condition

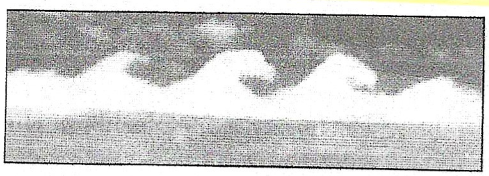


FIGURE 5-3
Kelvin–Helmholtz breaking waves outlined by a billow-cloud formation. [*John Davidson Photos/Alamy Stock Photo*].