Kelvin-Helmholtz Inviscid Shear Layer Instability

Realizable splitter plate with ΔU uniform streams, wherein perturbation increases downstream

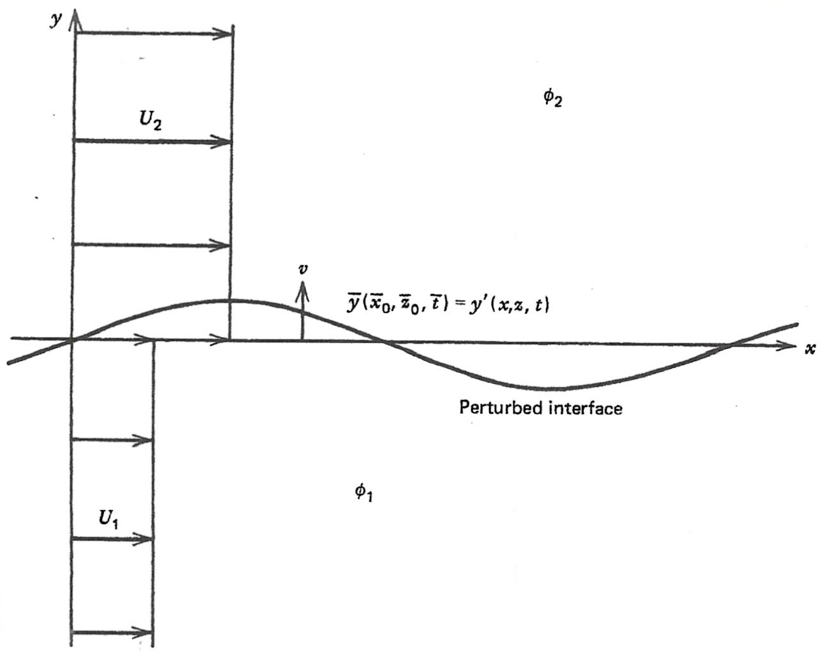

Figure 25.2 Shear layer nomenclature.

ie study of inviscid vortex sheet. Recall previous discussion 3(5) of viscous diffusion of vortex sheet:

$u = U\,\mathrm{erf}(\eta)$

$\eta = y/2\sqrt{\nu t}$, $\delta = 5.52\sqrt{\nu t}$

$U_1 = -U$

$U_2 = U$

$$\nabla^2\varphi_1 = 0 \qquad \nabla^2\varphi_2 = 0 \qquad \nabla\varphi_1 = U_1 \quad y \to -\infty$$

$$\nabla\varphi_2 = U_2 \quad y \to \infty$$

Kinematic free surface BC $\frac{DF}{Dt} = 0$ on $y = y'$ $\qquad F = y - f(x,z,t)$ surface function

$y' = f(x,z,t)$

$$\frac{\partial F}{\partial t} + \underline{V}\cdot\nabla F = 0 \qquad \underline{V}_1 = \nabla\varphi_1$$

$$= u_1\hat{i} + v_1\hat{j} + w_1\hat{k}$$

$$-y'_t + v_1 - u_1 y'_x - w_1 y'_z = 0 \qquad \nabla F = -y'_x\hat{i} + \hat{j} - y'_z\hat{k}$$

or $$v_1 = y'_t + u_1 y'_x + w_1 y'_z = \frac{\partial\varphi_1}{\partial y}$$

$y = y'$

Similarly for upper fluid $$v_2 = y'_t + u_2 y'_x + w_2 y'_z = \frac{\partial\varphi_2}{\partial y}$$

Dynamic Free Surface BC ie pressure is continuous across the interface

$$\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + P/\rho = C(t) \qquad \text{unsteady Bernoulli equation}$$

$$\phi_{1t} + \tfrac{1}{2}(\nabla\phi_1)^2 - C_1 = \phi_{2t} + \tfrac{1}{2}(\nabla\phi_2)^2 - C_2 \qquad \text{on } y = y'$$

Basic flow satisfies same problem on $y' = 0$

$$C_1 - \tfrac{1}{2}U_1^2 = C_2 - \tfrac{1}{2}U_2^2$$

Next introduce perturbations from the basic flow

$$\phi_1 = U_1 x + \phi_1' \qquad \nabla^2\phi_1' = 0 \qquad \nabla\phi_1' = 0 \quad y = -\infty$$
$$\phi_2 = U_2 x + \phi_2' \qquad \nabla^2\phi_2' = 0 \qquad \nabla\phi_2' = 0 \quad y = \infty$$

Kinematic BC is linearized and applied on $y = 0$

$$v_1' = y'_t + U_1 y'_x = \phi'_{1y}$$
$$v_2' = y'_t + U_2 y'_x = \phi'_{2y} \qquad \text{on } y = 0$$

Similarly dynamic BC

$$\phi'_{1t} + U_1\phi'_{1x} = \phi'_{2t} + U_2\phi'_{2x} \qquad \text{on } y = 0$$

Assume normal mode perturbation solutions

wave numbers

α = real

β = real

$\omega = 2\pi f = \alpha c_R$

$\lambda = 2\pi / k$

c_a = phase velocity $= \frac{\alpha c_R}{k} = \omega / k$

$= \lambda f$

for $\beta = 0$

$c_a = c_R$

$$\begin{Bmatrix} y' \\ \phi_1' \\ \phi_2' \end{Bmatrix} = \begin{Bmatrix} \hat{y} \\ \hat{\phi}_1(y) \\ \hat{\phi}_2(y) \end{Bmatrix} \exp\left[i(\alpha x + \beta z - \alpha c t)\right]$$

$\hat{y}$ = constant = original interface displacement (sign perturbed quantities)

$c = c_r + i c_i$ = complex wave speed

$c_i > 0$ unstable & grows exponentially in time

$-i\alpha c = -\alpha(i c_r - c_i)$

$= \alpha c_i - i\alpha c_r$

Substitution Laplace + ∞ BC

$$\hat{\phi}_1(y) = A_1 \exp(k y)$$
$$\hat{\phi}_2(y) = A_2 \exp(-k y)$$

$k = (\alpha^2 + \beta^2)^{1/2}$

$\underline{k} = (\alpha, 0, \beta)$

Substitution y', ϕ_1' & ϕ_2' in kinematic BC

$$A_1 = i\alpha \frac{\hat{y}}{k} (U_1 - c)$$

$$A_2 = -i\alpha \frac{\hat{y}}{k} (U_2 - c)$$

then in dynamic BC

$$(U_1 - c)^2 = -(U_2 - c)^2$$

$$c = c_R + i c_i = \tfrac{1}{2}(U_1 + U_2) \pm i \tfrac{1}{2}|U_2 - U_1|$$

+ sign unstable ALL wave numbers α, β

ie all shear layers $U_1 \neq U_2$ invisidly unstable all wave length disturbances!

growth rate is $\exp(\alpha c_I t)$ i.e. for waves with $k=\alpha$ will grow fastest with phase speed

$$c_{ph} = \frac{\alpha c_R}{k} = c_R = \frac{1}{2}(U_1+U_2)$$

disturbance travels at the average speed
base flow

Viscous effects that spread "diffuse" the velocity profile stabilize the flow for $\lambda \approx$ shear layer thickness, whereas longer waves governed by inviscid analysis

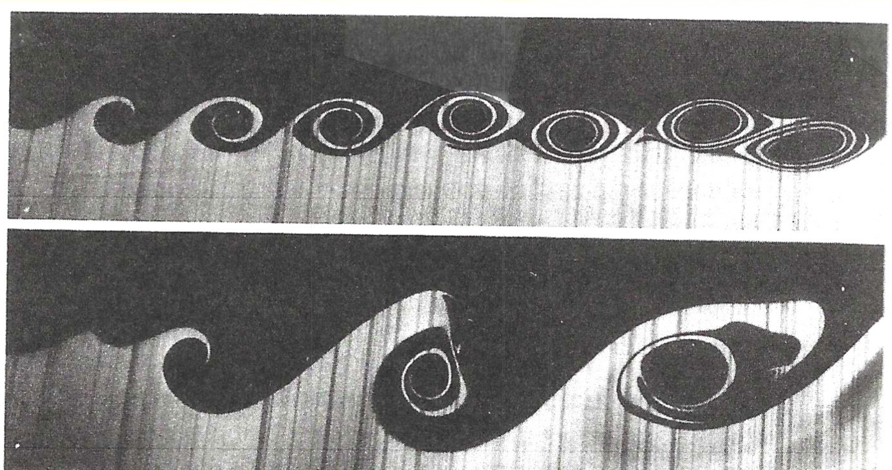

Figure 25.1 Kelvin–Helmholtz instability of a shear layer. The lower water stream, which contains a fluorescent dye, moves slower than the upper stream. A perturbation is introduced to initiate the growth in a regular pattern. Frequency is halved in the lower picture. Courtesy of F. A. Roberts, P. E. Dimotakis, and A. Roshko, California Institute of Technology.

$$\nabla^2 \phi_1' = 0 \qquad \phi_1' = \hat{\phi}_1(y)\, \exp^{\gamma} \qquad \gamma = i(\alpha x + \beta z - \alpha c t)$$

$$\phi_{1xx}' + \phi_{1yy}' + \phi_{1zz}' = 0$$

$$-\alpha^2 \hat{\phi}_1 - \beta^2 \hat{\phi}_1 + \hat{\phi}_{1yy} = 0 \quad \Rightarrow \quad \hat{\phi}_1(y) = A_1 e^{ky}$$

$$k = (\alpha^2 + \beta^2)^{1/2}$$

$$\phi_1' = A_1 e^{ky} e^{\gamma} \qquad -\alpha^2 A_1 - \beta^2 A_1 + k^2 A_1 = 0$$

$$\phi_2' = A_2 e^{-ky} e^{\gamma} \qquad \text{ie} \quad \hat{\phi}_2(y) = A_2 e^{-ky}$$

$$\phi_1 = U_1 x + A_1 e^{ky} e^{\gamma} \qquad \phi_2 = U_2 x + A_2 e^{-ky} e^{\gamma}$$

$$\eta' = \hat{\eta}\, e^{\gamma}$$

$$\phi_{1y}' = A_1 k\, e^{ky} e^{\gamma}$$

$$\eta_x' = \hat{\eta}\, e^{\gamma}(i\alpha) \qquad A_1 k e^{ky} e^{\gamma} = \hat{\eta}\, e^{\gamma}(-i\alpha c)$$

$$\eta_t' = \hat{\eta}\, e^{\gamma}(-i\alpha c) \qquad + U_1 \hat{\eta}\, e^{\gamma}(i\alpha)$$

$$y = 0 \quad A_1 k = \hat{\eta}(-i\alpha c) + U_1 \hat{\eta}(i\alpha)$$

$$A_1 = i\alpha \frac{\hat{\eta}}{k}(U_1 - c) \qquad A_2 = -i\alpha \frac{\hat{\eta}}{k}(U_2 - c)$$

$$A_1(-i\alpha c) + U_1 A_1 (i\alpha) = A_2(-i\alpha c) + U_2 A_2 (i\alpha)$$

$$A_1 (i\alpha)(U_1 - c) = A_2 (i\alpha)(U_2 - c)$$

$$(U_1 - c)^2 = -(U_2 - c)^2$$

$$U_1^2 - 2U_1 c + c^2 = -U_2^2 + 2U_2 c - c^2 \qquad ax^2 + bx + c = 0$$

$$2c^2 - 2(U_1 + U_2)c + U_1^2 + U_2^2 \qquad x = (-b \pm \sqrt{b^2 - 4ac})/2a$$

$$c = \underbrace{\tfrac{1}{2}(U_1 + U_2)}_{c_r} \pm \left[4(U_1 + U_2)^2 - 8(U_1^2 + U_2^2)\right]^{1/2} / 4$$

$c = c_r + i c_i$

$$4(U_1^2 + 2U_1 U_2 + U_2^2) - 8U_1^2 - 8U_2^2$$

$$-4U_1^2 + 8U_1 U_2 - 4U_2^2 = -4(U_1 - U_2)^2$$

$$\pm \tfrac{1}{2}[U_1 - U_2] = \pm i \underbrace{\tfrac{1}{2}|U_2 - U_1|}_{c_i}$$

+ sign unstable

− sign stable decay disturbance